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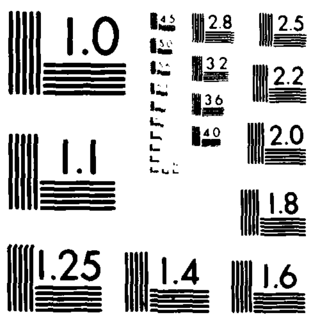

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VARIABLE-DIMENSION COMPLEXES WITH APPLICATIONS

by

Robert M. Freund

TECHNICAL REPORT SOL 80-11
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PREFACE

In the past several years, researchers in simplicial pivoting algorithms have developed a new class of algorithms, called variable-dimension algorithms. The dimension of the simplices encountered in the execution of these algorithms can vary significantly and it is this variability that sets these algorithms apart from other simplicial pivoting algorithms.

With the partial goal of presenting a unified view of these variable-dimension algorithms, we introduce a new mathematical structure called a V-complex (which is short for "variable-dimension complex"). A V-complex is axiomatically defined using elementary concepts from algebraic topology. With the addition of a labelling function on a V-complex, we develop a theory and a characterization of paths generated by V-complexes, that are reminiscent of the paths generated by typical simplicial pivoting algorithms.

For a given V-complex, we define its associated H-complex (which is short for "homogeneous-dimension complex"). We then show that an H-complex is an n-dimensional pseudomanifold; furthermore, path-following on V-complexes is equivalent to and is a "projection" of the well-known path-following scheme on a pseudomanifold, as applied to the H-complex.

Path-following on V-complexes gives rise to constructive proofs of a host of lemmas from combinatorial topology, including the Sperner Lemma, Scarf's Dual Sperner Lemma, Kuhn's Strong Cubical Sperner Lemma, and Tucker's Lemma on the n-cube. Exploiting the structure V-complexes, we

present two new combinatorial lemmas, the Generalized Sperner Lemma, and a new lemma on the n-cube. These combinatorial lemmas have interesting relationships with existence theorems for fixed-points, antipodal points, stationary points, nonlinear complementarity points, and equilibrium points in n-person non-cooperative game theory. The Generalized Sperner Lemma results in a new Covering Theorem on the simplex, as well as two other new results on the simplex.

V-complexes are also used to present a unifying theory of variable-dimension simplicial pivoting algorithms. We show that essentially all of these algorithms can be viewed as path-following schemes on a V-complex.

Similar to the orientation theory for pseudomanifolds, we present an orientation theory for H-complexes, which gives insight into the behavior of the paths on H- and V-complexes. We then give sufficient conditions on a V-complex that guarantee that the associated H-complex is orientable.

In Chapter I, we review some basic concepts from algebraic topology and triangulations. In Chapter II, we motivate our study by presenting Reiser's algorithm for the nonlinear complementarity problem, and van der Laan and Talman's first fixed-point algorithm. In Chapter III, we introduce and develop the theory of V-complexes and H-complexes. In Chapter IV, we present and give constructive proofs of the combinatorial lemmas noted above. In Chapter V, we show the relationship between these lemmas and certain existence theorems. We also prove new results using these lemmas. In Chapter VI, we present an orientation theory for H-complexes, and give sufficient conditions for the orientability of

H-complexes. In Chapter VII, we demonstrate that virtually all variable-dimension algorithms can be viewed in terms of a V-complex. In Chapter VIII, we make some concluding remarks and give suggestions for further research.

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NOTATION

$\mathbb{R}^n$	real n-dimensional space
$\mathbb{R}^{n+}$ or $\mathbb{R}^n_+$	$\{x \in \mathbb{R}^n \mid x \geq 0\}$
e	the vector of 1's, (1, 1, 1, ... , 1)
$\emptyset$	the empty set
$\triangle$	symmetric difference operator on sets; $S \triangle T = \{x \mid x \in S \cup T,\ x \notin S \cap T\}$
$\setminus$	difference operator on sets; $S \setminus T = \{x \mid x \in S,\ x \notin T\}$
$\lVert\cdot\rVert_2$	the Euclidean norm; $\lVert x\rVert_2 = \sqrt{x_1^2 + \cdots + x_n^2}$
$\lVert\cdot\rVert_\infty$	$\lVert x\rVert_\infty = \max\limits_i \lvert x_i\rvert$
Diameter of a simplex	Let σ be a real n-simplex. The diameter of σ is equal to $\max\limits_{x,y \in \sigma} \lVert x-y\rVert$, where $\lVert\cdot\rVert$ is any norm.
Diameter of a triangulation	Let C be a triangulation. The diameter of C is equal to $\sup\limits_{\sigma \in C}$ [diameter of σ].
$A_{\cdot i}$	the $i^{\underline{th}}$ column of a matrix A
$A_{i\cdot}$	the $i^{\underline{th}}$ row of a matrix A.
e^i	the $i^{\underline{th}}$ unit vector in $\mathbb{R}^n$.

CHAPTER I

COMPLEXES, PSEUDOMANIFOLDS, ORIENTATION, TRIANGULATIONS

1.0. Introduction

In this first chapter, we review some basic concepts from algebraic topology that provide a basis for the material which follows. Specifically, we discuss the notions of an abstract complex, pseudomanifolds, orientation, and triangulations. Most of this material is not new, but a review is helpful.

1.1. Complexes

An *abstract complex* consists of a set of vertices K^0 and a set of finite nonempty subsets of K^0, denoted K, such that

i) $v \in K^0$ implies $\{v\} \in K$

ii) $\emptyset \neq x \subset y \in K$ implies $x \in K$.

The elements of K are called *simplices*. Suppose $x \in K$ and $|x| = n+1$, where $|\cdot|$ denotes cardinality. Then x is called an *n-dimensional simplex*, or simply an *n-simplex*. Condition (i) above means that all members of K^0 are 0-simplices, and condition (ii) means that K is closed under subsets. Technically, an abstract complex is defined by the pair (K,K^0). However, since the set K^0 is implied by K, it is convenient to simply denote the complex by K alone.

As an example, consider

$$K = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,d\} \{d\}\}.$$

Then $K^0 = \{a,b,c,d\}$. $\{a,b\}$ is a 1-simplex. $\{a,b,c\}$ is a 2-simplex, and $\{d\}$ is a 0-simplex.

An abstract complex K is said to be finite if the set K^0 is finite. An abstract complex K is said to be locally finite if for each $v \in K^0$, the set of simplices containing v is a finite set. More formally, K is locally finite if and only if for each $v \in K^0$,

$$\{x \in K \mid v \in x\} \text{ is a finite set.}$$

Clearly, if K is finite, it is locally finite. The converse is not always true.

The previous example is a finite complex. As an example of a locally finite complex that is not finite, let

$$K = \{\{1\}, \{2\}, \{3\}, \ldots , \{1,2\}, \{2,3\}, \{3,4\}, \ldots \} .$$

As an example of a non-locally finite complex, let

$$K = \{\{1\}, \{2\}, \{3\}, \ldots , \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \ldots \}.$$

A subset L of K is said to be a subcomplex of K if L itself is a complex.

1.2. Pseudomanifolds

A particular class of complexes, called pseudomanifolds, is central to the theory to be developed. An n-dimensional pseudomanifold, or more simply an n-pseudomanifold, is a complex K such that

i) $x \in K$ implies there exists $y \in K$ with $|y| = n+1$, and $x \subset y$.

ii) If $x \in K$ and $|x| = n$, then there are at most two n-simplices that contain x.

Let

$$K = \{\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a\}, \{b\}, \{c\}, \{d\}\} .$$

Then K is a 2-dimensional pseudomanifold.

As an example of a complex which is not a pseudomanifold, let

$$K = \{\{a,b,c\}, \{a,b,d\}, \{a,b,e\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,d\}, \{b,d\} \{a,e\}, \{b,e\}, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}\} .$$

Here $\{a,b\}$ is contained in three 2-simplices.

Let K be an n-pseudomanifold. The <u>boundary</u> of K, denoted ∂K, is defined to be the set of simplices $x \in K$ such that x is contained in an (n-1)-simplex $y \in K$, and y is a subset of exactly one n-simplex of K.

Let $K = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{b,c\}, \{c,d\}\}$. Then $\partial K = \{\{a\}, \{d\}\}$.

Not all pseudomanifolds have a nonempty boundary. For example, let

$$K = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{b,c\}, \{c,d\}, \{d,a\}\} .$$

Then $\partial K = \emptyset$.

An n-pseudomanifold K is said to be homogeneous if for any pair of n-simplices $x, y \in K$, there is a finite sequence $x = x_1, x_2, x_3, \dots, x_m = y$ of n-simplices in K such that $x_i \cap x_{i+1}$ is an (n-1)-simplex in K, for $i = 1, \dots, m-1$. The preceding examples of pseudomanifolds are all homogeneous.

As an example of a non-homogeneous pseudomanifold, let

$$K = \{\{a,b,c\}, \{c,d,e\}, \{a,b\}, \{a,c\}, \{b,c\}, \{c,d\}, \{c,e\}, \{d,e\}$$
$$\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\} .$$

There is no "path" of n-simplices connecting $\{a,b,c\}$ and $\{c,d,e\}$.

Before closing this section, we define the notion of a pivot and a neighboring pair. Let K be an n-pseudomanifold and let x be an n-simplex in K. Let $x = \{v_0, \dots, v_n\}$. Let $y = \{v_1, \dots, v_n\}$. If $y \notin \partial K$, there is a unique $w \in K^0$ such that $\{w, v_1, \dots, v_n\}$ is an n-simplex in K. The process of exchanging v_0 for w to obtain a new n-simplex is called a pivot. In general, if x and z are n-simplices and z can be obtained from x by a pivot, x and z are said to be a neighboring pair, or simply neighbors.

1.3. Orientation

Let K be a homogeneous n-pseudomanifold, and let x be an n-simplex in K. Let $(v_0, \dots, v_n)$ be some fixed ordering of the vertices of x. For an arbitrary ordering $(v_{j_0}, \dots, v_{j_n})$ of x, this ordering is said to have a (+) orientation if and only if the permutation

$$(j_0, \dots, j_n)$$

is even; otherwise the orientation is (-).

Now let us extend this notion to all of K. Fix an ordering of all n-simplices of K. Let x be an n-simplex and let y be an n-simplex obtained by pivoting on an element v_{j_i} of x and replacing v_{j_i} by w. We say that the pair (x,y) is <u>coherently-oriented</u> if the orderings $(v_{j_0}, \ldots, v_{j_n})$ and $(v_{j_0}, \ldots, v_{j_{i-1}}, w, v_{j_{i+1}}, \ldots, v_{j_n})$ are differently oriented, i.e. one is (+) and the other is (-). K is said to be <u>orientable</u> if it is possible to specify orientations on all n-simplices of K in a way that all neighboring n-simplices x, y are coherently-oriented.

It is important to note a convenient way to go about orienting K, if K is orientable. Choose an n-simplex x and an ordering of its vertices and designate this ordering as (+). Then, by the homogeneity of K, we can orient all simplices that form a *neighboring* pair with x, and hence all n-simplices of K.

A natural question to ask at this point is whether or not all homogeneous pseudomanifolds are orientable. The answer is no. Figure 1.1 schematically represents a non-orientable 2-pseudomanifold that is equivalent to the famous Möbius strip. The maximal elements of K are $\{a,b,c\}$, $\{a,c,d\}$, $\{c,d,e\}$, $\{d,e,f\}$, $\{e,f,a\}$, and $\{f,a,b\}$. It is a simple exercise to verify that K cannot be oriented.

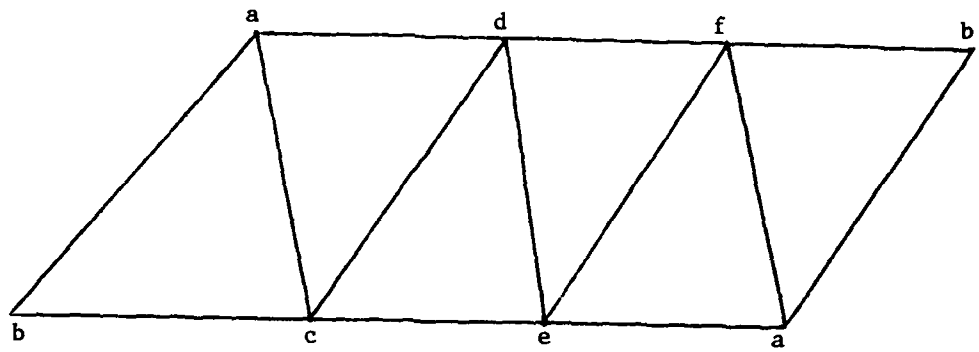


Figure 1.1

Figure 1.2, on the other hand, represents an orientable 2-pseudomanifold.

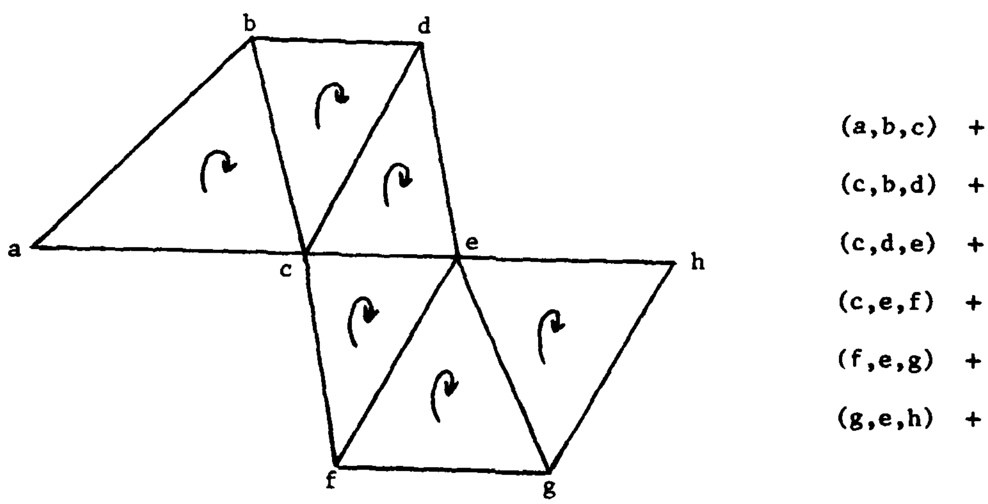


Figure 1.2

Finally, we define induced orientation on the boundary of K. Let K be a homogeneous orientable n-pseudomanifold such that ∂K is not empty. Let y be an (n-1)-simplex in ∂K. Then there is a unique n-simplex $x \in K$ such that $y \subset x$. Orient K coherently. Let $(v_{j_0}, \ldots, v_{j_n})$ be an ordering of the vertices of x. $y = x \setminus \{v_{j_i}\}$ for some i uniquely determined. Denote the orientation of the ordering $(v_{j_0}, \ldots, v_{j_n})$ by $Or(v_{j_0}, \ldots, v_{j_n})$. Then we define the <u>induced orientation</u> on y as

$$Or(v_{j_0}, \ldots, v_{j_{i-1}}, v_{j_{i+1}}, \ldots, v_{j_n}) = (-1)^i \, Or(v_{j_0}, \ldots, v_{j_n}) .$$

<u>Proposition 1.1</u>. Induced orientation is well-defined.

<u>PROOF</u>. Let y be an (n-1)-simplex in ∂K, and let x be the unique n-simplex in K that contains y. Let $(i_0, \ldots, i_{n-1})$ be an ordering of the vertices of y, and let $(j_0, \ldots, j_n)$ and $(\ell_0, \ldots, \ell_n)$ be orderings of the vertices of x, from which $(i_0, \ldots, i_{n-1})$ is derived. $y = x \setminus \bar{v}$ for some unique $\bar{v} \in x$. $\bar{v} = v_{j_r} = v_{\ell_s}$ for some unique r, s. If $r = s$, then $(j_0, \ldots, j_n) = (\ell_0, \ldots, \ell_n)$, and

$$Or(i_0, \ldots, i_{n-1}) = (-1)^r \, Or(j_0, \ldots, j_n) = (-1)^s \, Or(\ell_0, \ldots, \ell_n)$$

trivially.

So suppose $s > r$. It takes $s-r$ transpositions to change $(j_0, \ldots, j_n)$ to $(\ell_0, \ldots, \ell_n)$. Hence $(-1)^r \, Or(j_0, \ldots, j_n)$ $= (-1)^r (-1)^{s-r} \, Or(\ell_0, \ldots, \ell_n) = (-1)^s \, Or(\ell_0, \ldots, \ell_n)$. $\otimes$

1.4. Triangulations

The pseudomanifolds discussed so far are very abstract objects. They can be schematically represented, but trying to picture what they are all about is not easy. An n-dimensional pseudomanifold is an abstraction of a triangulation of an n-dimensional set in $\mathbb{R}^n$. The m-simplices of pseudomanifolds correspond to geometric objects, which by an unfortunate tradition, are also called m-simplices. In order to formally define a triangulation, we first need to define a real m-simplex in $\mathbb{R}^n$.

Let $v^0, \ldots, v^m$ be vectors in $\mathbb{R}^n$. $v^0, \ldots, v^m$ are said to be affinely independent if the matrix

$$\begin{bmatrix} v^0 & \cdots & v^m \\ 1 & \cdots & 1 \end{bmatrix}$$

has rank m+1. If $v^0, \ldots, v^m$ are affinely independent then their convex hull, denoted $\langle v^0, \ldots, v^m \rangle$ is said to be an m-dimensional simplex, or more simply an m-simplex. All m-simplices are closed and bounded polyhedral convex sets. Let $\{v^{j_0}, \ldots, v^{j_k}\}$ be a subset of $\{v^0, \ldots, v^m\}$. Then $\langle v^{i_0}, \ldots, v^{j_k}\rangle$ is called a k-dimensional face of k-face of $\langle v^0, \ldots, v^n \rangle$. Any k-face of $\langle v^0, \ldots, v^m \rangle$ is a k-simplex itself. An (m-1)-face of an m-simplex is called a facet of the m-simplex.

With the notions of simplices and faces in mind, we are ready to define a triangulation. Let H be an m-dimensional convex set in $\mathbb{R}^n$. Let C be a collection of m-simplices σ together with all of their faces. C is a triangulation of H if

i) $H = \bigcup_{\sigma \in C} \sigma$

ii) $\sigma, \tau \in C$ imply $\sigma \cap \tau \in C$

iii) If σ is an (m-1) simplex of C, σ is a face of at most two m-simplices of C.

The connection between triangulations and pseudomanifolds should be clear. Corresponding to each simplex σ in C is its set of vertices $\{v^0,\ldots,v^k\}$. Let K be the collection of these sets of vertices. Then K is an m-dimensional pseudomanifold.

Before concluding this section, we describe two useful triangulations, one of $\mathbb{R}^n$, and one of $T^n \triangleq \{x \in \mathbb{R}^n | e^T x = 1\}$.

1.4.1. Kuhn's triangulation of $\mathbb{R}^n$

Let Z^n denote the integral points in $\mathbb{R}^n$, let $N = \{1,\ldots,n\}$, and let π be a permutation of N. We define the simplex

$$\sigma(y^0, \pi) = \langle y^0, y^1, \ldots, y^n \rangle \quad \text{where} \quad y^i = y^{i-1} + e^{\pi_i}, \quad i = 1,\ldots, n,$$

where $y^0 \in Z^n$, and e^i is the $i^{\underline{th}}$ unit vector in $\mathbb{R}^n$. The collection of all such $\sigma(y,\pi)$ as y ranges over all of Z^n and π ranges over all permutations, together with all faces of $\sigma(y,\pi)$, is a triangulation of $\mathbb{R}^n$. By scaling these simplices, we obtain triangulations of $\mathbb{R}^n$ with arbitrarily small diameters of the simplices.

1.4.2. The Scarf-Hansen triangulation of S^n

Let c be a fixed positive integer. Let Q be the following $n \times n$ matrix:

$$Q = \frac{1}{c}\begin{bmatrix} -1 & 0 & \cdot & \cdot & \cdot & 0 & +1 \\ +1 & -1 & \cdot & & & & 0 \\ 0 & +1 & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & -1 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & +1 & -1 \end{bmatrix}$$

Let the $i^{\underline{th}}$ column of Q be denoted by q^i. Let π be a permutation of $N = \{1,\ldots,n\}$ with any one element of N missing. Finally, let y^0 be an element of Z^n such that $e^T y^0 = c$. Then we define the simplex

$$\sigma(y^0, \pi) = \langle \frac{y^0}{c}, y^1, \ldots, y^{n-1} \rangle ,$$

where $y^i = y^{i-1} + q^{\pi_i}$. The collection of all such $\sigma(y^0,\pi)$ together with all faces of $\sigma(y^0,\pi)$, is a triangulation of T^n. Let $S^n = \{x \in \mathbb{R}^n \,|\, e^T x = 1,\ x \geq 0\}$. Then the restriction of this triangulation to S^n is also a triangulation of S^n and all faces of S^n.

1.5. A Word on 0-Dimensional Pseudomanifolds

In the definition of a complex, the empty set $\emptyset$ is not an admissible element of the complex K. However, it is extremely desirable to allow, and insist, that the empty set be an element of K, if K is a

0-dimensional pseudomanifold. 0-dimensional pseudomanifolds either consist of one element or two elements. Hence $K = \{\{a\}\}$ or $K = \{\{a\}, \{b\}\}$. We shall allow the empty set to be the unique (-1)-dimensional simplex contained in K. If K has one element, then upon amending K, we have $K = \{\emptyset, \{a\}\}$, and we say $\emptyset \in \partial K$. If K has two elements, the amended K is $\{\emptyset, \{a\}, \{b\}\}$, and $\partial K = \emptyset$; hence $\emptyset \notin \partial K$.

1.6. References

Pertinent references for complexes and pseudomanifolds are Spanier [39] and Eaves [4]. Some of the material on orientation was taken from Lemke and Grotzinger [30]. The material on triangulations is based on Kuhn [21], Scarf [35], and Todd [41].

It should be noted that the notion of orienting pseudomanifolds can be extended to triangulations by the use of determinants. The interested reader can refer to Eaves [6] and Eaves and Scarf [8] for a development on orienting triangulations.

CHAPTER II

THE ALGORITHMS OF REISER AND VAN DER LAAN AND TALMAN

2.0 Introduction

In the past few year, researchers in mathematical programming have developed a new class of algorithms for computing fixed-points, solutions to the nonlinear complementarity problem, and equilibrium points in non-cooperative games. Unlike previous algorithms, these particular algorithms generate simplices of varying dimensions, and hence are called variable-dimension simplicial algorithms. As stated in the introduction, this dissertation presents a unifying view of these variable-dimension algorithms. In order to motivate the reader for the material of Chapter III and beyond, we present two of these algorithms in this chapter. The first algorithm, due to Reiser [33], is used to find an approximate solution to the nonlinear complementarity problem. The second algorithm, due to van der Laan and Talman [24], is used to find an approximate fixed point on the unit simplex.

The aim of this chapter is to illustrate certain properties of the paths generated by these algorithms. Hence, we only state the algorithms' pivot rules and present sample paths that each algorithm could generate. For complete descriptions of these algorithms, see the references above.

2.1. Reiser's Algorithm

Reiser's algorithm, *mutatis mutandis*, is based on Kuhn's triangulation (denoted T) of $\mathbb{R}^n$ scaled by a constant $c > 0$. Let w be

a fixed vertex of the triangulation. Let $L(\cdot)$ be a labelling function that assigns to each vertex of T an element of $\{\pm 1, \ldots, \pm n\}$. The algorithm is as follows:

Step 0 (Initialization). Set $v^1 = w$, $q = 1$ (q is the index of the vertex to be labelled), $m = 1$ (number of vertices of the actual simplex S). Go to Step 1.

Step 1 (Labelling). Let $\ell = L(v^q)$. If there exists a vertex v of S with $L(v) = -\ell$, go to Step 4. If there is a vertex v^k, $k \neq q$, of S with $L(v^k) = \ell$, go to Step 2. Otherwise go to Step 3.

Step 2 (Dropping or Replacing a Vertex). v^k is replaced by the only vertex $\bar{v}^k$ which can be added in order to obtain another $(m-1)$-simplex of T in the affine hull of S. If there is an $i \in \{1,\ldots,n\}$ with $(\bar{v}_i^k - w_i)(v_i^k - w_i) < 0$, determine that $j \in \{1,\ldots,m\}$ with $|L(v^j)| = i$, drop $\bar{v}^k$, let $k \leftarrow j$, $m \leftarrow m-1$, and go to Step 2. Otherwise let $v^k \leftarrow \bar{v}^k$, $q \leftarrow k$, and go to Step 1.

Step 3 (Adding a Vertex). Let $v^1 \leq \cdots \leq v^m$ be the vertices of S. If $\ell > 0$, let $v^{m+1} \leftarrow v^m + c \cdot e^{\ell}$. Otherwise, let $v^{m+1} \leftarrow v^1 - c \cdot e^{|\ell|}$. let $m \leftarrow m+1$, $q \leftarrow m$, and go to Step 1.

Step 4 (Termination). A simplex has been found with two vertices, v^{i_1}, v^{i_2}, such that $L(v^{i_1}) = -L(v^{i_2})$. Stop.

It can be proven that this algorithm can always be executed and that cycling cannot occur. Finite termination depends on regularity conditions imposed on $L(\cdot)$ either directly or through a function $f(\cdot)$ from which $L(\cdot)$ is derived.

Figure 2.1 illustrates a sample path that could be generated by the algorithm in $\mathbb{R}^2$. With the aid of this sample path, we make some observations about paths generated by the algorithm.

First, we see that the algorithm generates simplices of varying dimensions, but it does not do so arbitrarily! When the simplices are strictly inside a given quadrant (relative to w), it generates 2-simplices. When the algorithm is moving along a coordinate axis, it usually generates 1-simplices. Were we to examine a sample path generated in $\mathbb{R}^n$, the above remarks would generalize.

Second, the adjacency rules appear to be different for each quadrant. In the southeast quadrant, adjacent 2-simplices share the labels $\{1,-2\}$. In the southwest quadrant, adjacent 2-simplices share the labels $\{-1,-2\}$, etc. Along the coordinate axes, a related phenomenon takes place. In the north pointing axis, adjacent 1-simplices share the label $\{2\}$. In the south pointing axis, adjacent 1-simplices share the label $\{-2\}$, etc.

Finally, we remark that we have made no real use of the fact that the simplices of T have relative interiors. Indeed, we only use the vertices of the simplices, and the only extensively used property of T is its relation to a pseudomanifold.

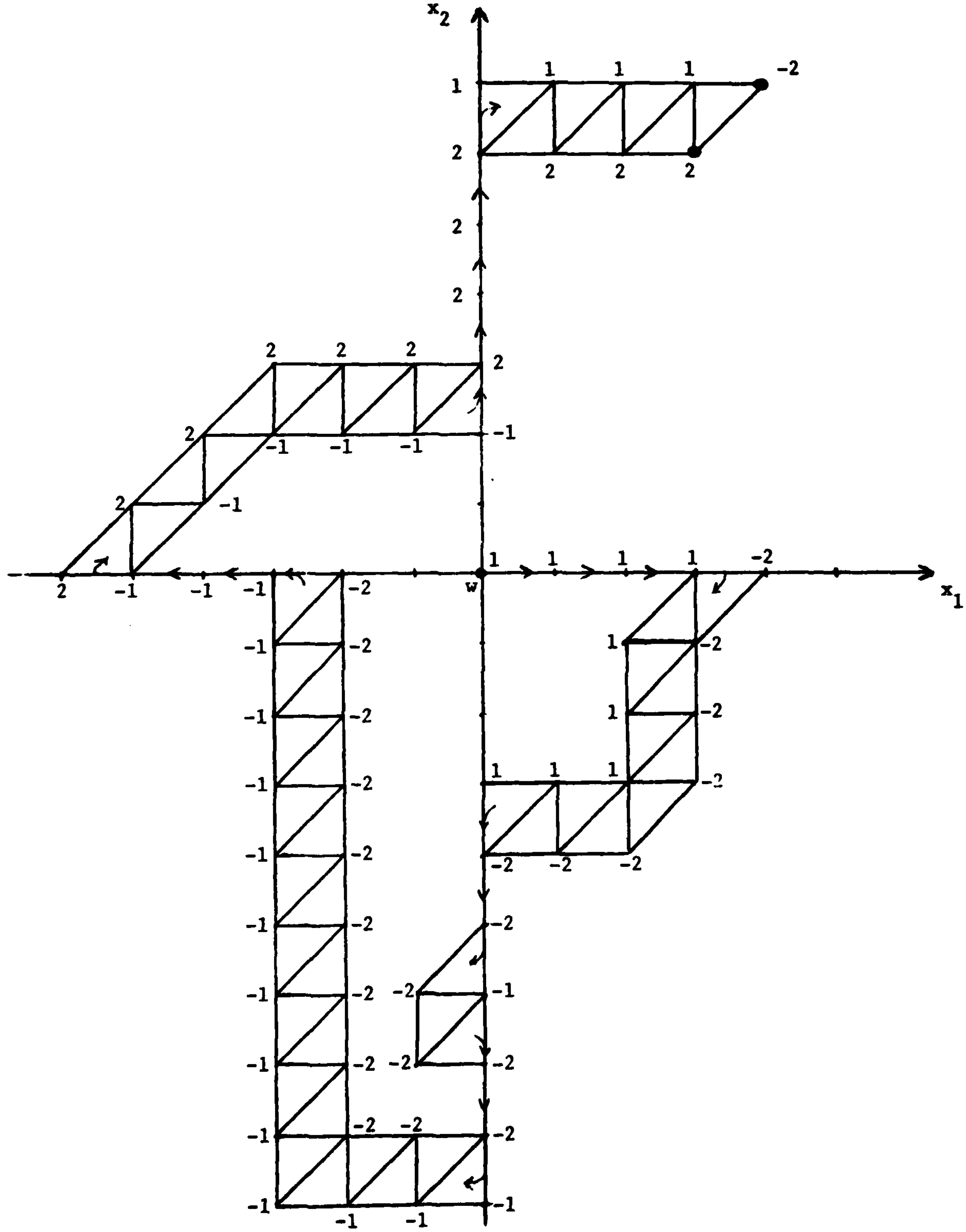


Figure 2.1

The above remarks suggest that we redefine adjacency between two simplices in terms of the region of $\mathbb{R}^n$ in which they are located. Furthermore, they suggest that we assign labels to entire regions of $\mathbb{R}^n$. We could (and later on we shall) refer to the northeast region of Figure 2.1 as the "$\{1,2\}$-region" and the north-pointing axis as the "$\{1\}$-region." Finally, the remarks suggest we look at the algorithm combinatorially, in terms related to pseudomanifolds.

2.2. van der Laan and Talman's First Algorithm

van der Laan and Talman's first algorithm for computing fixed points on the unit simplex uses Scarf's triangulation of S^n, which we denote by C. Let c be the scaling factor for the triangulation, and fix w, a vertex of C. Let $L(\cdot)$ be a labelling function that assigns to each vertex of C an element of $\{1,\ldots,n\}$. van der Laan and Talman's algorithm appears below. In the notation of the algorithm, $N = \{1,\ldots,n\}$ and $T \subset N$. γ^T is a permutation of the elements of T used to define the simplex. t is the dimension of the current simplex under scrutiny, q is the index of the incoming vertex, and k is the index of the outgoing vertex. R is a work vector, $R \in \mathbb{R}^n$. $q(i)$ is the $i^{\underline{th}}$ column of Q defined in Chapter 1, section 4.2. A simplex encountered in the algorithm is described by v^0, T, γ^T, where for $i = 1,\ldots,t$, $v^i = v^{i-1} + q(\gamma_i^T)$.

<u>Step 0</u> (Initialization). Set $T \leftarrow \emptyset$, $\gamma^T \leftarrow \emptyset$, $t \leftarrow 0$, $R \leftarrow 0$, $v^0 \leftarrow w$, $q \leftarrow 0$.

Step 1 (Labelling a Vertex). Set $\ell = L(v^q)$. If $\{L(v^0),\ldots, L(v^t)\} = \{1,\ldots,n\}$, go to Step 4. If ℓ is different from the labels of all other vertices, go to Step 3. Otherwise $\ell = L(v^k)$ for some $k \in \{0,\ldots,t\}\setminus\{q\}$. Go to Step 2.

Present value of k	New value of v^0	New value of γ^T	R	New q
$k = 0$	$v^0 + q(\gamma_1^T)$	$(\gamma_2^T,\ldots,\gamma_t^T,\gamma_1^T)$	$R + e(\gamma_1^T)$	t
$1 \le k \le t-1$	v^0	$(\gamma_1^T,\ldots,\gamma_{k-1}^T,\gamma_{k+1}^T,\gamma_k^T,\ldots,\gamma_t^T)$	R	k
$k = t$	$v^0 - q(\gamma_t^T)$	$(\gamma_t^T,\gamma_1^T,\ldots,\gamma_{t-1}^T)$	$R - e(\gamma_t^T)$	0

Table 2.1

Step 2 (Replacing or Deleting a Vertex). Determine new values of v^0, γ^T, R, and q from Table 2.1. If $R \geq 0$, go to Step 1. Otherwise, set k equal to the index j such that $L(v^j) = \gamma_t^T$. Set $R \leftarrow R + e(\gamma_t^T)$, $T \leftarrow T\setminus\{\gamma_t^T\}$, $\gamma^T \leftarrow (\gamma_1^T,\ldots,\gamma_{t-1}^T)$, $t \leftarrow t-1$. Go to Step 2.

Step 3 (Adding a Vertex). Set $v^{t+1} = v^t + q(\ell)$. Set $T \leftarrow T \cup \{\ell\}$, $\gamma^T = (\gamma_1^T,\ldots,\gamma_t^T,\ell)$, $t \leftarrow t+1$. $q \leftarrow t$. Go to Step 1.

Step 4 (Termination). A simplex has been found with $\{L(v^0),\ldots, L(v^t)\} = \{1,\ldots,n\}$, and hence $t = n-1$. Stop.

Under suitable conditions imposed on $L(\cdot)$ (i.e. that $L(\cdot)$ be a "proper" labelling, see Chapter IV, Section 1), it can be proven that the steps of the algorithm generate a unique finite sequence of simplices, and that the last simplex of the sequence satisfies the termination conditions of Step 4.

A sample path that the algorithm could generate in S^3 appears in Figure 2.2. As in the case of Reiser's algorithm, we see that in different regions of the simplex, the adjacency properties of the generated simplices varies. In the upper right region of the simplex, adjacent 2-simplices share the labels $\{1,2\}$, in the central left region of the simplex, adjacent 2-simplices share the labels $\{2,3\}$. Along the right-pointing axis from w, adjacent 1-simplices share the label $\{1\}$, etc. And, as in Reiser's algorithm, we only take advantage of the pseudo-manifold properties of the triangulation in the execution of the algorithm.

2.3. Conclusion and Final Remarks

We have seen that the variable-dimension algorithms of Reiser and van der Laan-Talman have a number of properties in common. As it turns out, numerous other variable-dimension algorithms (i.e., all such that have come to my attention) share these properties. In the next chapter, we shall define a special complex which has the above-mentioned properties. And we shall show in Chapters IV and VII that the variable-dimensional simplicial algorithms can be viewed as acting on specific realizations of this complex.

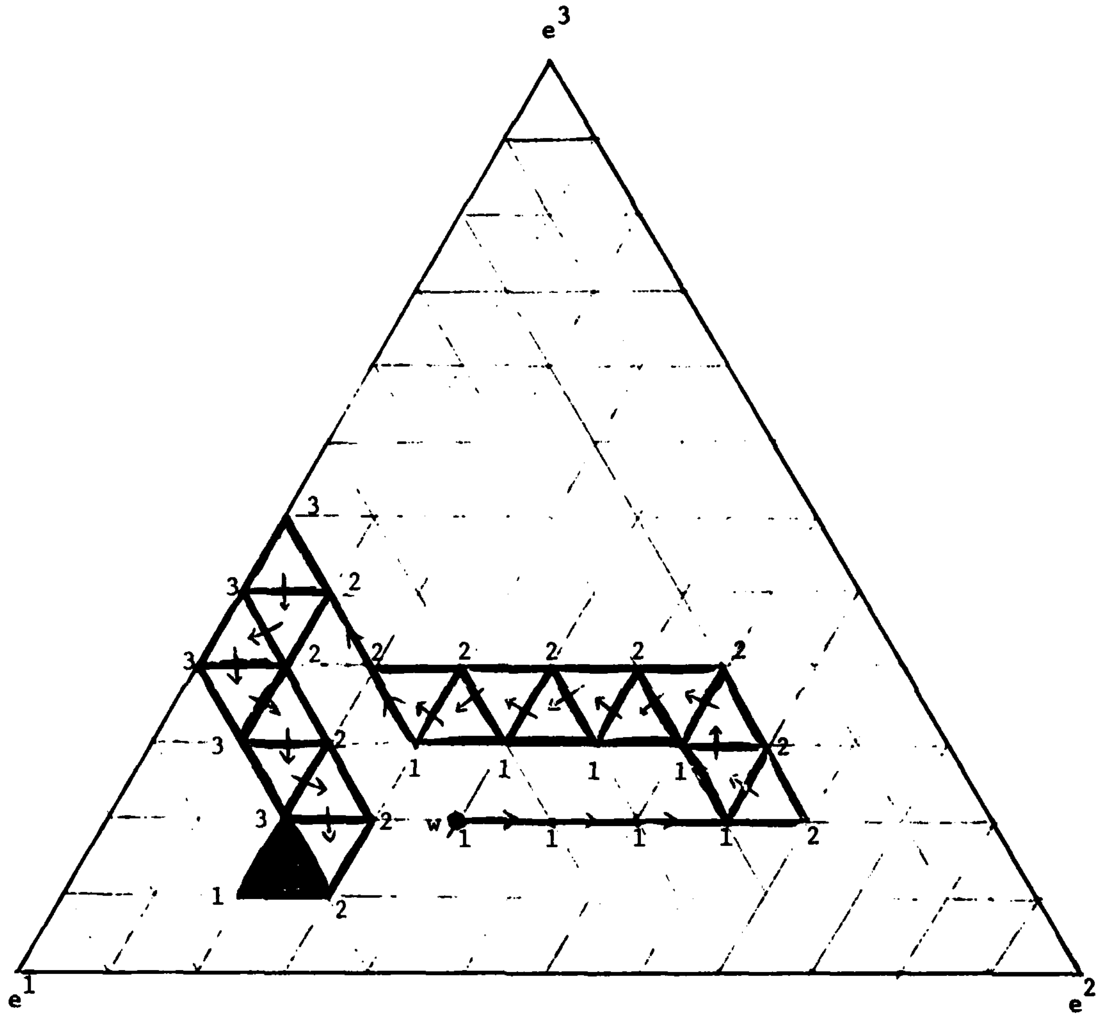


Figure 2.2

CHAPTER III

V-COMPLEXES AND H-COMPLEXES

3.0. Introduction

In this chapter, we define a particular type of complex, called a V-complex. We define adjacency between two simplices of a V-complex in a general manner, and then proceed to characterize paths generated by the adjacency properties of V-complexes. The net result is a path following scheme, or algorithm, that encompasses a variety of known algorithms, and lays a foundation for new algorithms as well.

The concept of a V-complex and the adjacency rule for simplices are the central theme of this dissertation.

The development of this chapter is fairly abstract, and directly uses the material of Chapter I.

3.1. V-Complexes

Let K be a simplicial complex with vertices K^0. Let N be a fixed finite nonempty set, which we call the label set. Let $\mathfrak{S}$ denote a collection of subsets of N, which we call the admissible subsets of N. Let $A(\cdot)$ be a set-to-set map, $A:\mathfrak{S} \to 2^K \setminus \{\emptyset\}$, where 2^S denotes the collection of subsets of a set S. K is said to be a V-complex if the following eight conditions are met:

i) K is a complex with vertices K^0

ii) $\Im \subset 2^N$

iii) $T \in \Im$, $S \in \Im$ implies $S \cap T \in \Im$

iv) $A(\cdot): \Im \to 2^K \setminus \{\emptyset\}$

v) For any $x \in K$, there is a $T \in \Im$ such that $x \in A(T)$

vi) For any $S, T \in \Im$, $A(S \cap T) = A(S) \cap A(T)$

vii) For $T \in \Im$, $A(T)$ is a subcomplex of K and is a pseudommanifold of dimension $|T|$, where $|\cdot|$ denotes the cardinality of the set.

viii) $T \in \Im$, $T \cup \{j\} \in \Im$, $j \notin T$ implies $A(T) \subset \partial A(T \cup \{j\})$.

Let us examine these properties. (i), (ii), and (iv) reiterate what has been said in the preceding paragraph. (iii) imposes some structure on $\Im$, namely that it is closed under intersections. (v) states that the map $A(\cdot)$ covers all simplices of K. (vi) states that $A(\cdot)$ is a homomorphism with respect to intersections. (vii) states that each $A(T)$ is an appropriately-dimensioned pseudomanifold. Condition (viii) stipulates how the pseudomanifolds $A(T)$ are arranged relative to each other, namely that $A(T)$ is part of the boundary of $A(T \cup \{j\})$.

As an example of a V-complex, consider Kuhn's subdivision of $\mathbb{R}^2$, as is used in Reiser's algorithm. Let $N = \{\pm 1, \pm 2\}$. Let $\Im$ be the collection of sets $\{1\}$, $\{-1\}$, $\{2\}$, $\{-2\}$, $\{1,2\}$, $\{1,-2\}$, $\{-1,2\}$, $\{-1,-2\}$ and $\emptyset$. Let K be the complex (actually a pseudomanifold itself) associated with Kuhn's triangulation. Then for each $T \in \Im$, we define

$$A(T) = \{x \in K \mid v \in x \text{ implies } i \cdot v_{|i|} \geq 0 \text{ for each } i \in T, \text{and } v_i = 0 \text{ if } i \notin T \text{ and } -i \notin T\}.$$

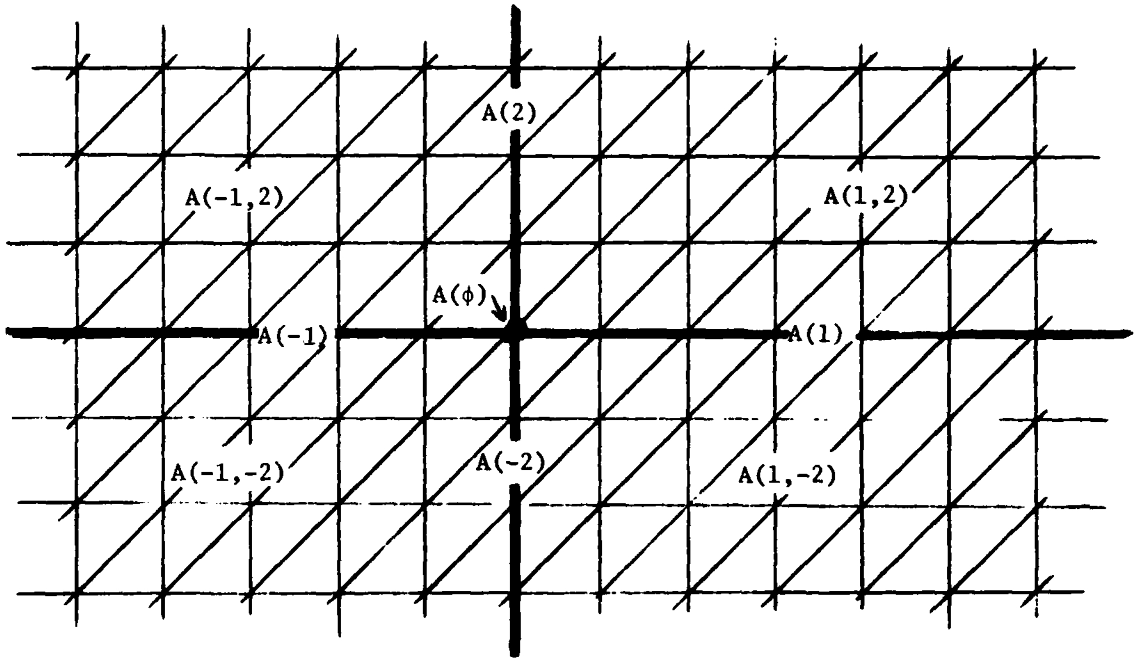


Figure 3.1

Figure 3.1 illustrates this V-complex. Note that for matters of convenience, the set brackets { } have been deleted. In the figure, $A(\emptyset)$ is the origin, $A(i)$ corresponds to one of the four axes emanating from the origin, and $A(i,j)$ corresponds to one of the four quadrants.

Figure 3.1 is by no means the only V-complex associated with $\mathbb{R}^2$. Figures 3.2 and 3.3 demonstrate other V-complexes associated with $\mathbb{R}^2$, with the triangulations omitted.

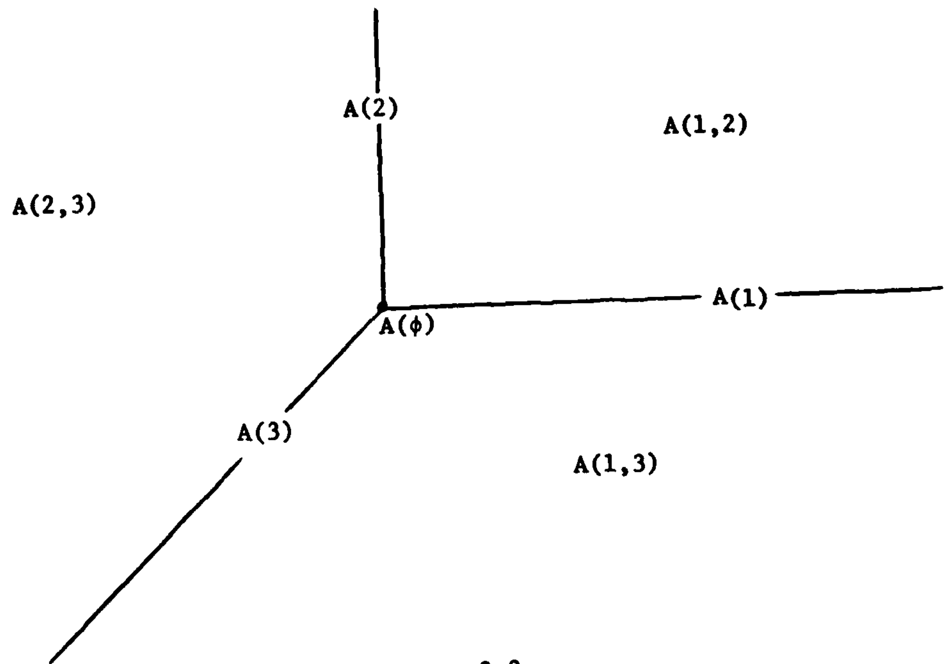


Figure 3.2

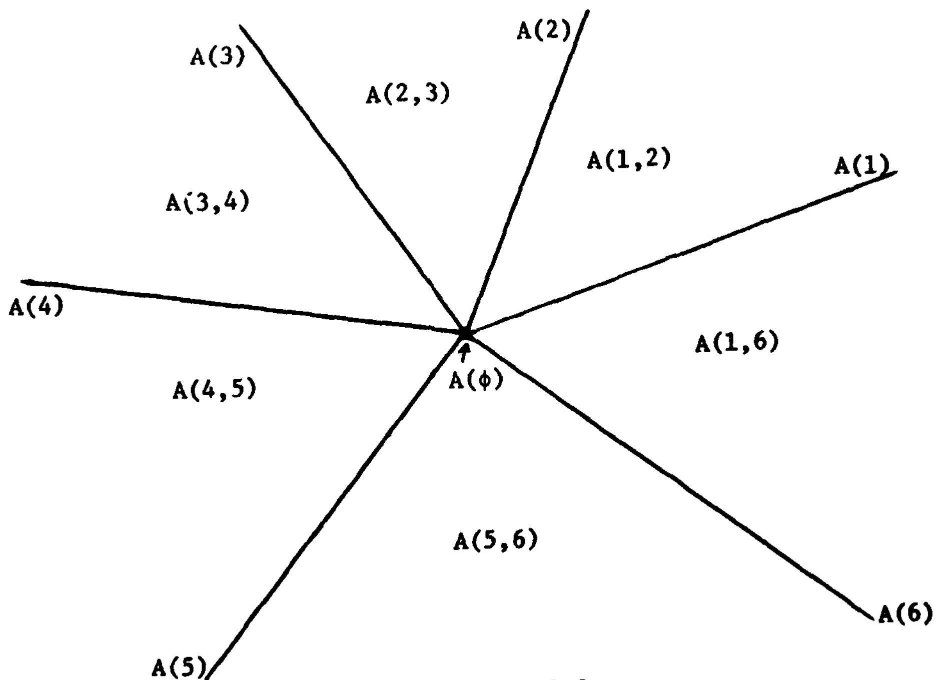


Figure 3.3

Suppose K is a V-complex. Let $x \in K$. We define

$$T_x = \bigcap_{\substack{T \in \mathfrak{I} \\ x \in A(T)}} T$$

T_x then is the smallest set T such that $x \in A(T)$. We say x is <u>full</u> if $|x| = |T_x| + 1$. x is a <u>full simplex</u> if it is a maximum-dimension simplex in $A(T_x)$.

For each $T \in \mathfrak{I}$, we also define $\partial' A(T)$ as

$$\partial' A(T) = \{x \in \partial A(T) \,|\, T_x = T\} .$$

We illustrate the above definitions in the V-complex in Figure 3.4. In the figure, the left-most vertex of the 2-simplex is $A(\emptyset)$, the "bottom" line segment is $A(1)$, the left-sided line segment is $A(2)$, and the simplex itself is $A(1,2)$.

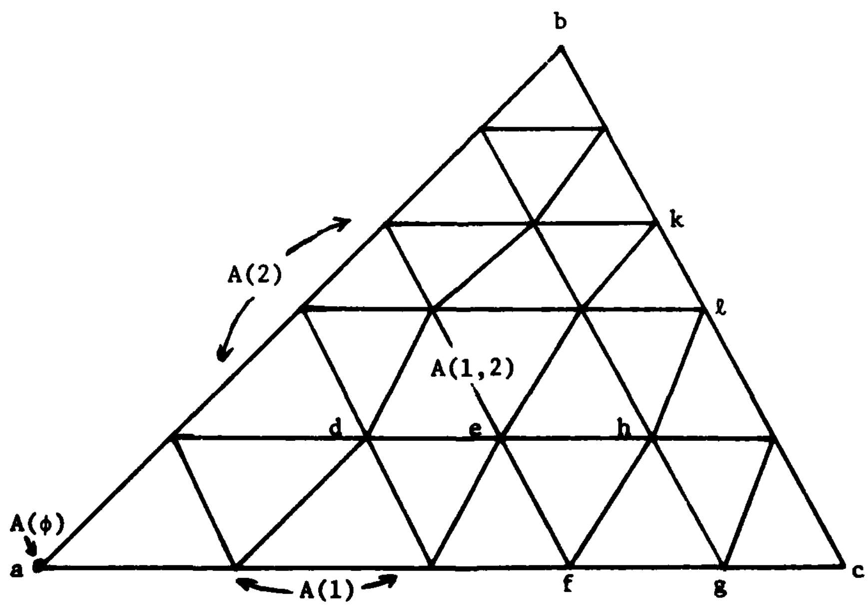


Figure 3.4

For $x = \{d,e\}$, $T_x = \{1,2\}$, for $x = \{f,g\}$, $T_x = \{1\}$. For $x = \{e,f,h\}$, $T_x = \{1,2\}$. The simplices $\{a\}$, $\{f,g\}$, and $\{e,h,f\}$ are all full. We have $\partial' A(1) = \{c\}$, $\partial' A(2) = \{b\}$, and $\partial' A(1,2)$ is the pseudomanifold corresponding to the line segment from b to c. Thus, while both $\{k,\ell\}$ and $\{f,g\}$ are elements of $\partial A(1,2)$, $\{k,\ell\} \in \partial' A(1,2)$, whereas $\{f,g\} \notin \partial' A(1,2)$.

When $T = \emptyset$, and $A(T)$ contains only one vertex, we define $\partial' A(\emptyset) = \{\emptyset\}$, so in this case $\partial' A(\emptyset) = \partial A(\emptyset) = \{\emptyset\}$. Thus, in Figure 3.4, $\partial' A(\emptyset) = \{\emptyset\}$.

3.2. Labelling Vertices and Adjacency on V-Complexes

Let K be a V-complex with label set N. Let $L(\cdot)$ be a function that assigns to each $v \in K^0$ an element $i \in N$. Such a function $L(\cdot)$ is a labelling function. For a simplex $x = \{v^0,\ldots,v^m\} \in K$, we define $L(x) = \cup_{v \in x} L(v)$. $L(x)$ is the set of labels spanned by the elements of x.

We define two distinct simplices $x, y \in K$ to be adjacent (written $x \sim y$) if

i) x and y are full

and ii) $L(x \cap y) = T_x \cup T_y$.

Note that adjacency is symmetric: $x \sim y$ if and only if $y \sim x$.

Figure 3.5 represents a V-complex whose vertices K^0 have labels $(\cdot)$. In the figure, we have the following adjacent simplices:

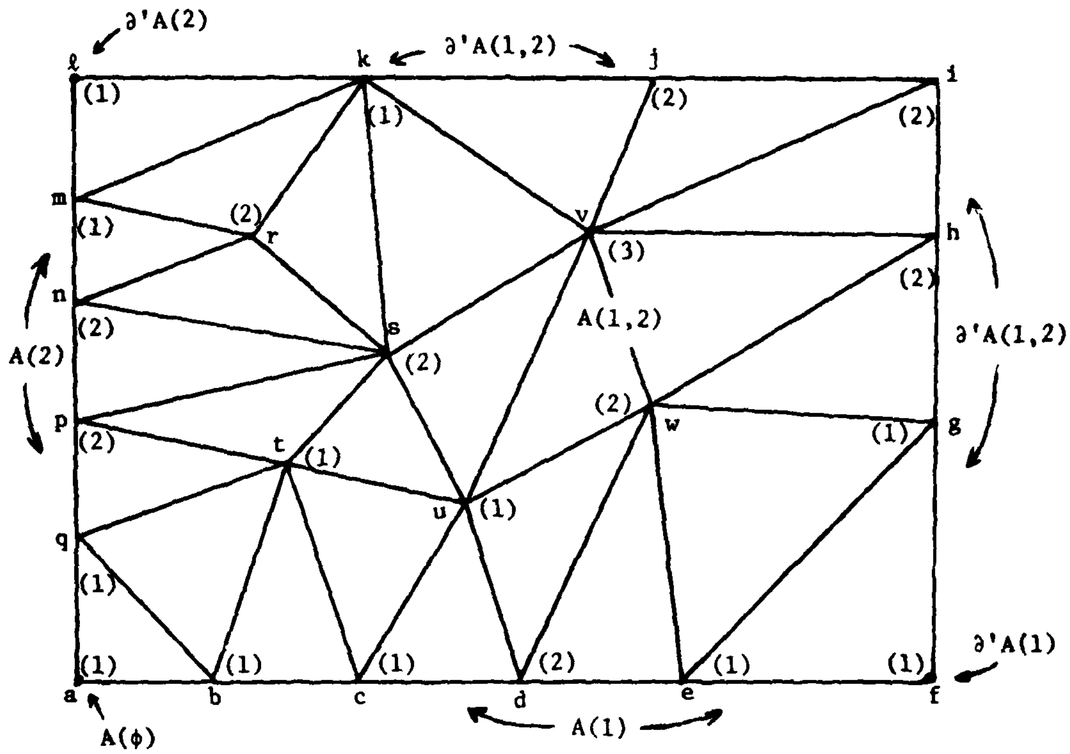


Figure 3.5

$\{a\}\sim\{a,b\}\sim\{b,c\}\sim\{c,d\}\sim\{c,d,u\}\sim\{u,d,w\}\sim\{u,w,v\}$

$\{u,s,v\}\sim\{s,u,t\}\sim\{t,s,p\}\sim\{p,t,q\}\sim\{p,q\}\sim\{p,n\}\sim\{n,m\}\sim\{n,m,r\}\sim\{m,r,k\}\sim\{k,r,s\}\sim\{s,k,v\}$

$\{h,g,w\}\sim\{g,w,e\}\sim\{w,e,d\}\sim\{d,e\}\sim\{e,f\}$.

Observe that, in the figure, any full simplex is adjacent to at most two other simplices. We shall see later on that this is true in general. Observe also that the adjacency relationship results in the formation of three distinct "paths" of simplices, each path being a string of simplices adjacent to one another.

The purpose of the remainder of this chapter is to give a characterization of these paths. However, we must develop the theory of V-complexes further before a complete characterization is possible.

3.3. H-Complexes

Let K be a V-complex with label set N and admissible sets $\mathcal{S}$. We wish to "lift" K into a pseudomanifold of dimension n where $n = |N|$. Without loss of generality, assume $N = \{1,\ldots,n\}$. Let K^0 be the set of vertices of K. We define artifical vertices $q_1,\ldots,q_n$. Let $\bar{Q} = \{q_1,\ldots,q_n\}$. Define $\bar{K}^0 = K^0 \cup \bar{Q}$. Define $Q_T = \{q_i \in \bar{Q} | i \in N \setminus T\}$. We now define:

$$\bar{K} = \{x \cup Q | x \cup Q \neq \emptyset,\ x \in K,\ Q \subset Q_{T_x}\}.$$

We have the following:

Theorem 3.1. $\bar{K}$ is an n-dimensional pseudomanifold.

PROOF. Clearly $\bar{K}$ is closed under nonempty subsets, and so is a complex. Let $x \cup Q \in \bar{K}$. Then there exists $y \subset A(T_x)$ that is full and $y \supset x$.

Let $P = Q_{T_x}$. Then we have $x \cup Q \subset y \cup P \in \bar{K}$. Furthermore,

$$|y \cup P| = |y| + |P| = |y| + n - |T_x| = |y| + n - (|y|-1) = n+1 .$$

Therefore every simplex in $\bar{K}$ is a subset of an n-simplex in $\bar{K}$. And clearly, $\bar{K}$ is closed under nonempty subsets. It only remains to show that each (n-1)-simplex of $\bar{K}$ is contained in at most two n-simplices.

Let $\bar{x} = x \cup Q_x$ be an n-simplex in $\bar{K}$, and let $\tilde{y} \subset \bar{x}$ be an (n-1)-simplex in $\bar{K}$. Suppose $\tilde{y} \subset \bar{z} \neq \bar{x}$, and $\bar{z}$ is an n-simplex in $\bar{K}$. We aim to show that $\bar{z}$ is uniquely determined by $\bar{x}$ and $\tilde{y}$. Since $\bar{x}$ is an n-simplex, x is full and $Q_x = Q_{T_x}$. We have three cases:

<u>Case 1.</u> $\tilde{y} = \bar{x} \setminus \{q_i\}$ for some $q_i \in Q_x$. Let $\bar{z} = z \cup Q_z$. If $z = x$, then $Q_z = Q_x$, and so $\bar{z} = \bar{x}$, a contradiction. Therefore $z \neq x$. But since $z \supset x$, we must have $z = x \cup \{w\}$ for some w. Therefore $Q_z = Q_x \setminus \{q_i\}$, and so $T_z = T_x \cup \{i\}$. By property (viii) of V-complexes, the choice of w, and hence $\bar{z}$, is unique.

<u>Case 2.</u> $y = \bar{x} \setminus \{v\}$ for some $v \in x$, and $x \setminus \{v\}$ is not full. We can write $\tilde{y} = y \cup Q_x$ where $y = x \setminus \{v\}$. Since y is not full, we must have $z = y \cup \{w\}$ for some $w \in K^0$, $w \notin y$, and hence $Q_z = Q_x$, whence $T_z = T_x$. The choice of w is uniquely determined, since $A(T_x)$ is a pseudomanifold.

Case 3. $\tilde{y} = \bar{x} \setminus \{v\}$ for some $v \in x$, and $x \setminus \{v\}$ is full. Again we write $\tilde{y} = y \cup Q_x$, where $y = x \setminus \{v\}$. Since y is full, $T_y = T_x \setminus \{i\}$ for some $i \in T_x$, and by property (viii) of V-complexes, $y \in \partial A(T_x)$. Hence we cannot have $z = y \cup \{w\}$ for any $w \in K^0$. Therefore the unique n-simplex of $\bar{K}$ containing $\tilde{y}$ is $\bar{z} = y \cup Q_x \cup \{q_i\}$. $\otimes$

We illustrate this result in Figures 3.6, 3.7, and 3.8.

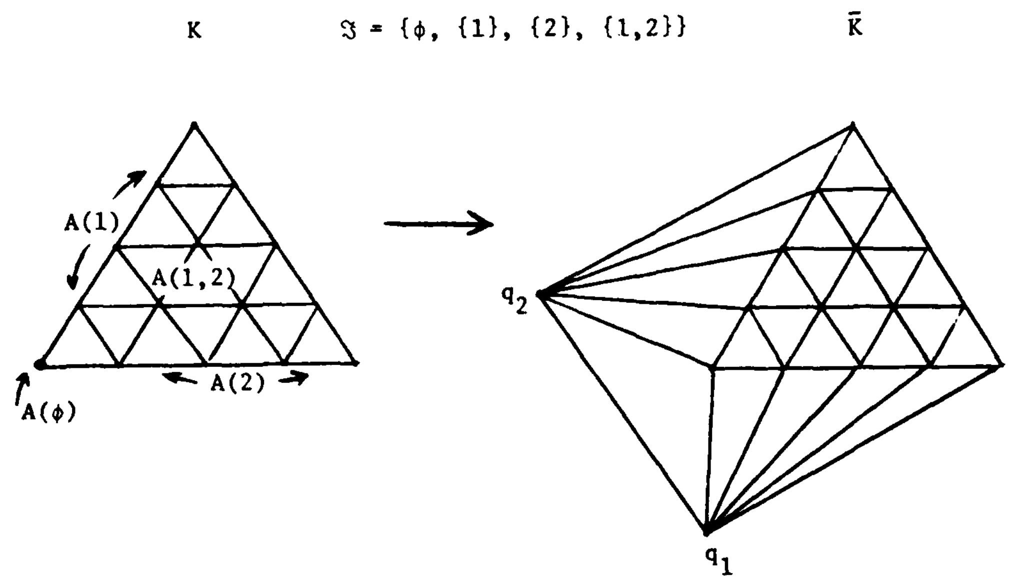


Figure 3.6

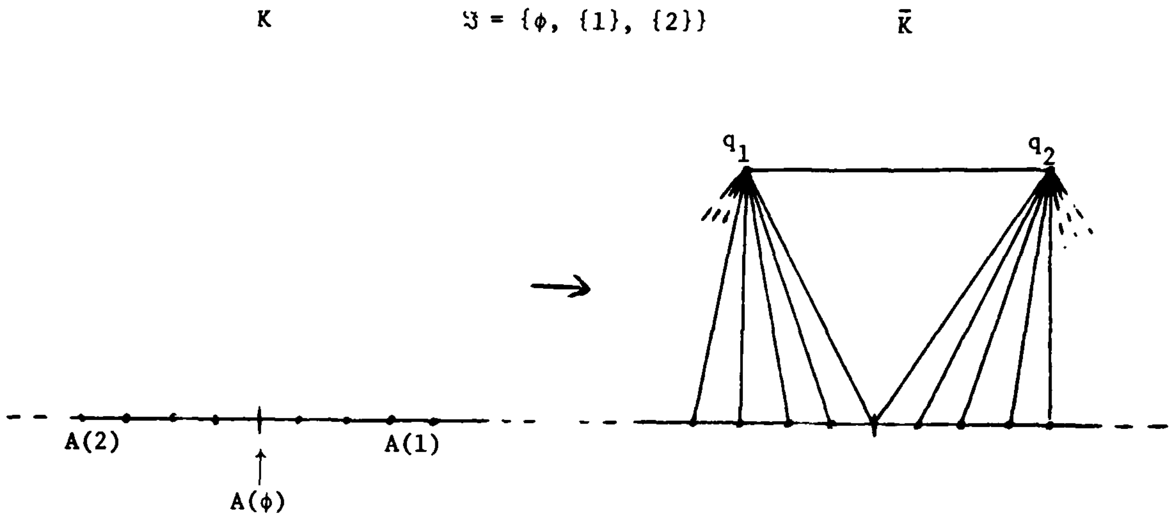


Figure 3.7

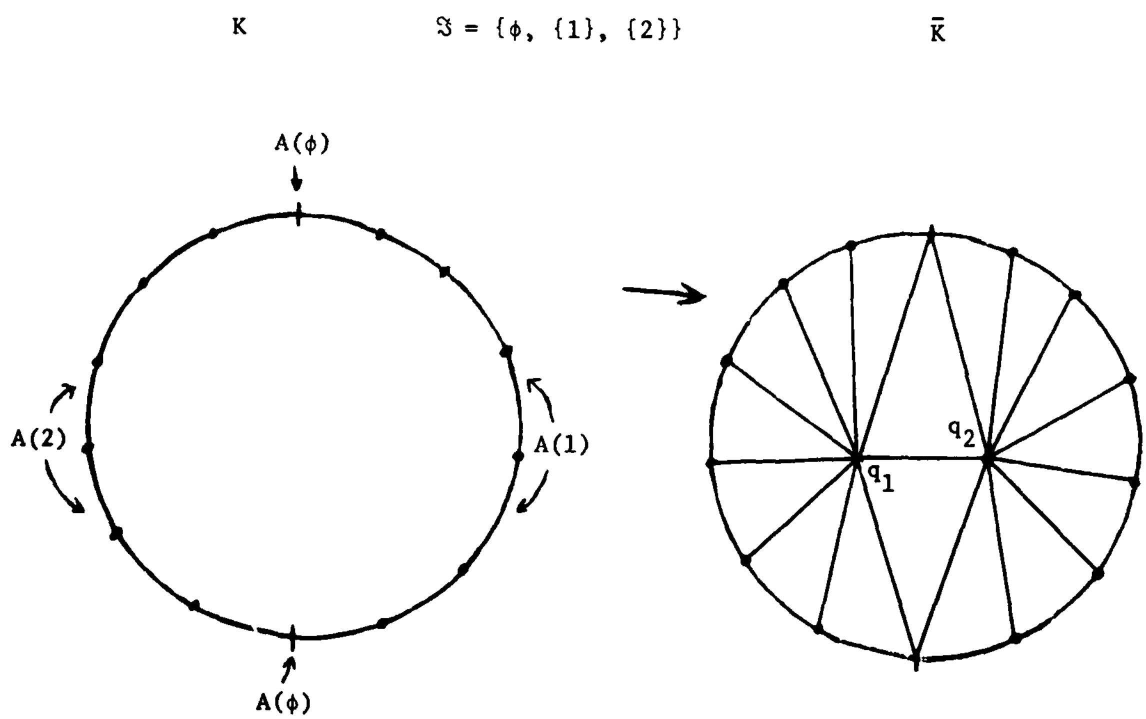


Figure 3.8

In Figure 3.8, $A(\emptyset)$ consists of the north and south "poles" of the circle and $A(1)$, $A(2)$ are the right and left arcs, respectively.

Our next task is a characterization of the boundary of $\bar{K}$.

<u>Theorem 3.2</u>. $\partial\bar{K} = S_1 \cup S_2$, where

$$S_1 = \{y \cup Q_y \in \bar{K} | y \in \partial' A(T_y)\}, \text{ and}$$

$$S_2 = \{y \cup Q_y \in \bar{K} | N \setminus \{i | q_i \in Q_y\} \notin \mathfrak{I}\}.$$

<u>PROOF</u>. We first prove that $S_1 \subset \partial\bar{K}$. Let $\bar{y} = y \cup Q_y$ be a maximal element of S_1. Then $y \in \partial' A(T_y)$ and $Q_y = Q_{T_y}$. The only n-simplex that contains $\bar{y}$ is of the form $y \cup \{v\} \cup Q_y$, where v is uniquely determined since $y \in \partial' A(T_y)$. Therefore $S_1 \subset \partial\bar{K}$.

Next we prove that $S_2 \subset \partial\bar{K}$. Let $\bar{y} = y \cup Q_y$ be a maximal element of S_2. Then y is full and $Q_y = Q_{T_y} \setminus \{q_i\}$ for some i, where $T_y \cup \{i\} \notin \mathfrak{I}$. Let $\bar{x} = \bar{y} \cup \{\alpha\}$ be an n-simplex in $\bar{K}$. We need to show α is uniquely determined. We cannot have $\alpha \in K^0$, since the set $T_y \cup \{i\} \notin \mathfrak{I}$. Hence $\alpha = q_j$ for some j. Suppose $j \neq i$. Then $T_y \cup \{i\} \setminus \{j\} \in \mathfrak{I}$, and in fact $T_y = T_y \cup \{i\} \setminus \{j\}$, whereby $j = i$, a contradiction. Therefore $\alpha = q_i$, and so $S_2 \subset \partial\bar{K}$. Therefore $S_1 \cup S_2 \subset \partial\bar{K}$.

Now let us prove the converse. Let $\bar{x} = x \cup Q_{T_x}$ be an n-simplex in $\bar{K}$ and let $\bar{y} \subset \bar{x}$ be an element of $\partial\bar{K}$, where $\bar{y} = y \cup Q$. We have two cases:

Case 1. $x = y \cup \{v\}$. Clearly we must have $y \in \partial A(T_x)$. If y were full, then $\bar{y} \subset y \cup Q_{T_y}$, and so $y \notin \partial\bar{K}$. Therefore y is not full. Hence $\bar{y} \in S_1$.

Case 2. $Q_{T_x} = Q \cup \{q_j\}$ for some $q_j \notin Q$. Suppose $N \setminus \{i | q_i \in Q\} \in \mathfrak{I}$. This means that $T_x \cup \{j\} \in \mathfrak{I}$. But then, by property (viii) of V-complexes, there is a unique $v \in K^0$ s.t. $y \cup \{v\} \in A(T_x \cup \{j\})$. Hence $\bar{y} \subset y \cup \{v\} \cup Q \in \bar{K}$, and hence $\bar{y} \notin \partial\bar{K}$, a contradiction. Therefore $N \setminus \{i | q_i \in Q\} \notin \mathfrak{I}$, and $\bar{y} \in S_2$.

Therefore $\partial\bar{K} \subset S_1 \cup S_2$, so $\partial\bar{K} = S_1 \cup S_2$. $\otimes$

3.4. Labelling Vertices and Adjacency on H-Complexes

Let K be a V-complex and $\bar{K}$ its associated H-complex. Let $L(\cdot): K^0 \to N$ be a labelling function on K^0. We extend $L(\cdot)$ to $\bar{K}^0$ by the simple rule that $L(q_i) = i$ for each $q_i \in \bar{Q}$, thereby obtaining a labelling function on $\bar{K}^0$. Let $\bar{x}$ be a simplex in $\bar{K}$. We define $L(\bar{x}) = \cup_{v \in \bar{x}} L(v)$.

We define two distinct n-simplices $\bar{x}, \bar{y} \in \bar{K}$ to be adjacent (written $\bar{x} \sim \bar{y}$) if

i) $\bar{x}$ and $\bar{y}$ are n-simplices

and ii) $L(\bar{x} \cap \bar{y}) = N$.

The above definition of adjacency is quite standard for labelling functions on pseudomanifolds (see Gould and Tolle [14] or Lemke and Grotzinger [30]). Note that if $\bar{x} \sim \bar{y}$, $\bar{x}$ and $\bar{y}$ must be neighbors.

3.5. Characterization of Paths on H-Complexes

Let K be a V-complex, $\bar{K}$ its associated H-complex, and let $L(\cdot)$ be a labelling function on K, extended to $\bar{K}$. The following theorem, whose proof we omit, follows from the standard "ghost story" argument of complementary pivot theory (see Lemke [29], Gould and Tolle [14], Kuhn [22], Eaves [5], or Scarf [34]).

Theorem 3.3. Let $\bar{x}$ be an n-simplex of $\bar{K}$. Then $\bar{x}$ is adjacent to at most two other n-simplices of $\bar{K}$. If $\bar{x}$ is adjacent to only one n-simplex of $\bar{K}$, then there is a unique (n-1)-simplex $\bar{y} \subset \bar{x}$ such that $L(\bar{y}) = N$ and $\bar{y} \in \partial\bar{K}$. $\otimes$

We define $\bar{B} = \{\bar{x} \in S_1 | L(\bar{x}) = N\}$ and $\bar{G} = \{\bar{x} \in S_2 | L(\bar{x}) = N\}$.

Proposition. $\bar{B} \cap \bar{G} = \emptyset$.

PROOF. Suppose $\bar{x} \in \bar{G}$. Then $|\bar{x}| = n$, so $\bar{x}$ is a maximal element of S_2. We can write $\bar{x} = x \cup Q$, where x is full. But then $\bar{x} \notin S_1$, since otherwise x is not full. Therefore $\bar{x} \notin \bar{B}$. $\otimes$

With the help of Theorem 3.3, we can construct and characterize "paths" on $\bar{K}$. Let $\langle \bar{x}_i \rangle_i$ be a maximal sequence of n-simplices of $\bar{K}$ such that $L(\bar{x}_i) = N$, $\bar{x}_i \sim \bar{x}_{i+1}$ and $x_{i-1} \neq x_{i+1}$ for any i. If $\bar{x}_k$ is a right-endpoint of the sequence, define $\bar{x}_{k+1}$ to be the unique subset of $\bar{x}_k$ such that $L(\bar{x}_{k+1}) = N$ and $\bar{x}_{k+1} \in \partial\bar{K}$. If $\bar{x}_k$ is a left-endpoint of the sequence, define $\bar{x}_{k-1}$ to be the unique subset of $\bar{x}_k$ such that $L(\bar{x}_{k-1}) = N$ and $\bar{x}_{k-1} \in \partial\bar{K}$. The new sequence, with possible endpoints added, is called a path on $\bar{K}$. Note that endpoints are elements of $\bar{G} \cup \bar{B}$.

We can characterize paths on $\bar{K}$ as one of six types.

Type I. $\langle \bar{x}_i \rangle_i$ where the sequence has no endpoints, and

i) $\bar{x}_i \sim \bar{x}_{i+1}$ for all $-\infty < i < +\infty$

ii) $\bar{x}_i \neq \bar{x}_j$ for any $i \neq j$.

Type II. $\langle \bar{x}_i \rangle_i$ where the sequence has no endpoints, and

i) $\bar{x}_i \sim \bar{x}_{i+1}$ for all $-\infty < i < +\infty$

ii) $\bar{x}_{i-1} \neq \bar{x}_{i+1}$ for all $-\infty < i < +\infty$

iii) There is an $m > 2$ such that $\bar{x}_i = \bar{x}_{i+m}$ for all $-\infty < i < +\infty$

iv) $\bar{x}_i \neq \bar{x}_{i+k}$ for any $0 < k < m$.

Type III. $\langle \bar{x}_i \rangle_i$ where the sequence consists of only three elements, say $\bar{x}_0$, $\bar{x}_1$; $\bar{x}_2$, and

i) $\bar{x}_0, \bar{x}_2 \in \bar{G} \cup \bar{B}$, $\bar{x}_0 \neq \bar{x}_2$

ii) $L(\bar{x}_1) = N$

iii) $\bar{x}_1$ is an n-simplex and $\bar{x}_0, \bar{x}_2 \subset \bar{x}_1$.

Type IV. $\langle \bar{x}_i \rangle_i$ has more than three elements, and has two endpoints, say $\bar{x}_0$ and $\bar{x}_m$, and

i) $\bar{x}_0, \bar{x}_m \in \bar{G} \cup \bar{B}$, and $\bar{x}_0 \neq \bar{x}_m$

ii) $\bar{x}_i \sim \bar{x}_{i+1}$ for all $0 < i < m-1$

iii) $\bar{x}_i \neq \bar{x}_j$ for any $i \neq j$, $0 \leq i, j \leq m$.

Type V. $\langle \bar{x}_i \rangle_i$ has only a left endpoint, say $\bar{x}_0$, and

i) $\bar{x}_0 \in \bar{G} \cup \bar{B}$

ii) $\bar{x}_i \sim \bar{x}_{i+1}$ for all $i > 0$

iii) $\bar{x}_i \neq \bar{x}_j$ for all $i, j \geq 0$, $i \neq j$.

Type VI. $\langle \bar{x}_i \rangle_i$ has only a right endpoint, say $\bar{x}_m$, and

i) $\bar{x}_m \in \bar{G} \cup \bar{B}$

ii) $\bar{x}_{i-1} \sim \bar{x}_i$ for all $i < m$

iii) $\bar{x}_i \neq \bar{x}_j$ for all $i, j \leq m$, $i \neq j$.

A type I path stretches infinitely in both directions. A type II path is a loop. A type III path is a "degenerate" path consisting of one n-simplex and two of its (n-1) subsimplices. A type IV path is a path with two endpoints. A Type V or Type VI path consists of one endpoint and stretches infinitely in one direction. A Type III path is illustrated in Figure 3.9.

In the applications of V-complexes and H-complexes, it is the endpoints of paths that are of interest. We have the following lemmas.

Lemma 3.4. Let $\bar{x} \in \bar{K}$. Then $\bar{x}$ is an endpoint of a path if and only if $\bar{x} \in \bar{G} \cup \bar{B}$.

PROOF. If $\bar{x}$ is an endpoint of a path, by definition $\bar{x} \in \bar{G} \cup \bar{B}$. Conversely, let $\bar{x} \in \bar{G} \cup \bar{B}$. There is a unique n-simplex $\bar{z} = \bar{x} \cup \{\alpha\}$ for some $\alpha \in \bar{K}^0$, and $L(\bar{z}) = N$. We can construct a path starting at $\bar{x} = x_0$, $\bar{x}_1 = \bar{z}$, etc. ⊗

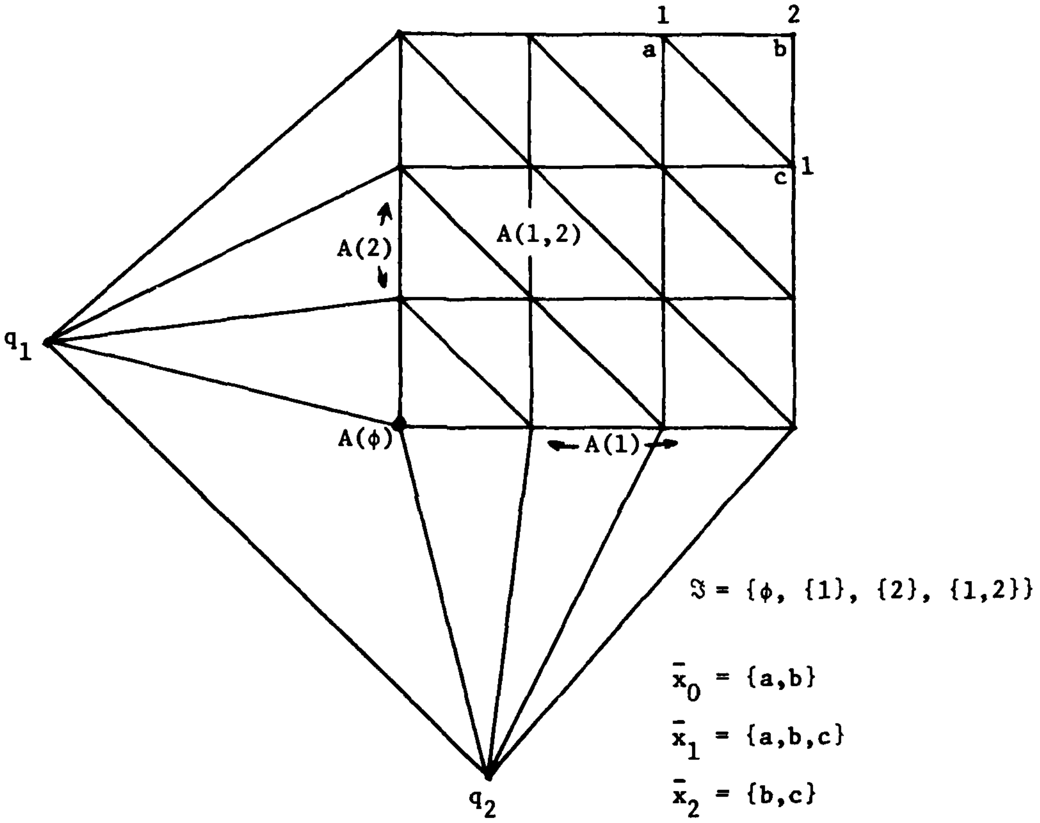

1
2
a
b
c
1
A(2)
A(1,2)
q_1
A(φ)
A(1)
q_2
$\mathfrak{I} = \{\phi, \{1\}, \{2\}, \{1,2\}\}$
$\bar{x}_0 = \{a,b\}$
$\bar{x}_1 = \{a,b,c\}$
$\bar{x}_2 = \{b,c\}$

Figure 3.9

Corollary 3.5. If $\bar{K}$ is finite, $\bar{B}$ and $\bar{G}$ have the same parity.

PROOF. If $\bar{K}$ is finite, the total number of endpoints of paths is finite and even. Each endpoint is in exactly one of the two sets above; hence, they have the same parity. ⊗

3.6. Characterization of Paths on V-Complexes

The characterization of paths on V-complexes is achieved by establishing a certain equivalence relationships between V-complexes and H-complexes. The first equivalence is given in the following lemma.

Lemma 3.6. Let $\bar{x}$ and $\bar{y}$ be n-simplices on $\bar{K}$. Let $\bar{x} = x \cup Q_x$, $\bar{y} = y \cup Q_y$. Then $\bar{x} \sim \bar{y}$ if and only if $x \sim y$.

PROOF. Suppose $\bar{x} \sim \bar{y}$. This means $L(\bar{x} \cap \bar{y}) = N$. We have

$$N = L(\bar{x} \cap \bar{y}) = L(x \cap y) \cup L(Q_x \cap Q_y) = L(x \cap y) \cup ((N \setminus T_x) \cap (N \setminus T_y)).$$

Therefore

$$L(x \cap y) = N \setminus ((N \setminus T_x) \cap (N \setminus T_y)) = N \setminus (N \setminus (T_x \cup T_y)) = T_x \cup T_y.$$

Thus we see that $L(x \cap y) = T_x \cup T_y$, and so $x \sim y$. The same argument in reverse shows that if $x \sim y$, then $\bar{x} \sim \bar{y}$. ⊗

Define $G = \{x \in K | x$ is full, $L(x) \supset T_x$, and $L(x) \not\in \mathfrak{I}\}$. G can be thought of as the goal set, for in most applications of V-complexes, the algorithm searches for an element of G.

Define $B = \{x \in K | x \in \partial' A(T_x)$, and $L(x) = T_x\}$. We have the following lemmas:

Lemma 3.7. Let $x \in K$, and let $\bar{x} = x \cup Q_{T_x}$. Then $x \in B$ if and only if $\bar{x} \in \bar{B}$.

PROOF. Let $x \in B$. $L(x) = T_x$. $L(\bar{x}) = L(x) \cup L(Q_{T_x}) = T_x \cup (N \setminus T_x) = N$. Furthermore, $x \in \partial' A(T_x)$, so $\bar{x} \in S_2$. Therefore $\bar{x} \in \bar{B}$.

Conversely let $\bar{x} \in \bar{B}$. Then $x \in \partial' A(T_x)$ and $L(x) = N \setminus L(Q_{T_x}) = N \setminus (N \setminus T_x) = T_x$, whence $x \in B$. ⊗

Lemma 3.8. Let $x \in K$ and $L(x) \supset T_x$, and let $\bar{x} = x \cup Q_{L(x)}$. Then $x \in G$ if and only if $\bar{x} \in \bar{G}$.

PROOF. Let $x \in G$. Then $L(x) = T_x \cup \{j\}$ for some $j \notin T_x$, where $T_x \cup \{j\} \notin \mathfrak{J}$. $Q_{L(x)} = Q_{T_x} \setminus \{q_j\} \subset Q_{T_x}$, so $\bar{x} \in \bar{K}$. Also, $N \setminus \{i \mid q_i \in Q_{L(x)}\} = N \setminus (N \setminus (T_x \cup \{j\})) = T_x \cup \{j\} \notin \mathfrak{J}$. Furthermore, $L(\bar{x}) = L(x) \cup L(Q_{L(x)}) = L(x) \cup (N \setminus (L(x)) = N$. Therefore $\bar{x} \in \bar{G}$.

Conversely, let $\bar{x} \in \bar{G}$. Then $L(x) = N \setminus \{i \mid q_i \in Q_{L(x)}\} \notin \mathfrak{J}$. Hence $x \in G$. ⊗

Let $x \in K$ be full. We define the degree of x, written $\deg(x)$, to be the number of distinct simplices of K adjacent to x.

Lemma 3.9. For any $x \in K$, $\deg(x) \leq 2$.

PROOF. From Lemma 3.6, we have $x \sim y$ if and only if $\bar{x} \sim \bar{y}$, where $\bar{x} = x \cup Q_{T_x}$ and $\bar{y} = y \cup Q_{T_y}$. Since $\bar{x}$ is adjacent to at most two simplices, so is x. ⊗

With the help of Lemma 3.9, we can construct paths on K. Let $\langle x_i \rangle_i$ be a maximal sequence of full simplices in K such that $x_i \sim x_{i+1}$, $x_{i-1} \not\sim x_{i+1}$, and $L(x_i) \supset T_{x_i}$. (Note that if $x_i \sim x_{i+1}$ then $L(x_i) \supset T_{x_i}$.) Let x_k be a left endpoint of the sequence. Then $\bar{x}_k$ is a left endpoint of the associated sequence in $\bar{K}$. Define $\bar{x}_{k-1}$ as in Section 3.5, and define x_{k-1} such that $\bar{x}_{k-1} = x_{k-1} \cup Q$ for appropriate $Q \subset \bar{Q}$. Likewise, if x_k is a right endpoint, define x_{k+1} analogously. The new sequence, with possible endpoints added, is a path on K.

We have the following characterization of endpoints of paths on K.

Lemma 3.10. Let $x \in K$. Then x is an endpoint of a path on K if and only if $x \in G \cup B$.

PROOF. Let x be an endpoint of a path on K. Then $\bar{x} = x \cup Q$ (for appropriate choice of $Q \subset \bar{Q}$) is an endpoint of a path on $\bar{K}$. $\bar{x} \in \bar{G} \cup \bar{B}$. So $x \in G \cup B$. ⊗

Lemma 3.11. If K is finite, B and G have the same parity.

PROOF. B and G, by definition, have no simplices in common. There is a one-to-one correspondence between elements of B (G) and elements of $\bar{B}$ ($\bar{G}$). Also, if K is finite, so is $\bar{K}$. Thus, by Corollary 3.5, B and G have the same parity. ⊗

We thus see a complete equivalence between paths on K and on $\bar{K}$. Hence we can classify paths on K as one of six types.

Type I. $\langle x_i \rangle_i$, where the sequence has no endpoints, and

i) $x_i \sim x_{i+1}$ for all i

ii) $x_i \neq x_j$ for all $i \neq j$.

Type II. $\langle x_i \rangle_i$, where the sequence has no endpoints, and

i) $x_i \sim x_{i+1}$ for all i

ii) $x_{i-1} \neq x_{i+1}$ for all i

iii) There is an $m > 2$ such that $x_i = x_{i+m}$ for all i

iv) $x_i \neq x_{i+k}$ for all i, all $0 < k < m$.

Type III. $\langle x_i \rangle_i$, where the sequence consists of only three elements, say x_0, x_1, x_2, and

i) $x_0, x_2 \in G \cup B$

ii) $L(x_1) \subset T_x$

iii) x_1 is full and $x_0, x_2 \subset x_1$.

Type IV. $\langle x_i \rangle_i$ has more than three elements, and has two endpoints, say x_0 and x_m, and

i) $x_0, x_m \subset G \cup B$ and $x_0 \neq x_m$

ii) $x_i \sim x_{i+1}$ for all $0 < i < m-1$

iii) $x_i \neq x_j$ for any $i \neq j$, $0 \leq i, j \leq m$.

Type V. $\langle x_i \rangle_i$ has only a left endpoint, say x_0, and

i) $x_0 \in G \cup B$

ii) $x_i \sim x_{i+1}$ for all $i > 0$

iii) $x_i \neq x_j$ for any $i, j \geq 0$, $i \neq j$.

Type VI. $\langle x_i \rangle_i$ has only a right endpoint, say x_m, and

i) $x_m \in G \cup B$

ii) $x_{i-1} \sim x_i$ for all $i < m$

iii) $x_i \neq x_j$ for all $i, j \leq m$, $i \neq j$.

3.7. The Algorithmic Development

There are two ways to develop an algorithm based on a V-complex and a labelling function $L(\cdot)$, depending on the nature of the set $A(\emptyset)$.

If $A(\emptyset)$ consists of a single 0-simplex, say $\{w\}$ and the empty set $\emptyset$, then $\emptyset \in B$, since $\emptyset \in \partial' A(\emptyset)$ and $L(\emptyset) = \emptyset = T_\emptyset$. Thus our algorithm consists of following a path whose endpoint is $\emptyset$.

If $A(\emptyset)$ consists of two 0-simplices, say $\{v\}$ and $\{w\}$, we have $v \sim w$, since $L(\{v\} \cap \{w\}) = L(\emptyset) = \emptyset = \emptyset \cup \emptyset = T_{\{v\}} \cup T_{\{w\}}$. Thus the algorithm consists of following the path containing $\{v\}$ and $\{w\}$ in one or both directions.

3.8. Concluding Remarks

The purpose of this chapter has been to show how to construct and follow paths on V-complexes. We used the construction of an H-complex to expedite the development of the theory. However, path following on H-complexes is an "ordinary" phenomenon familiar to researchers in

complement ry pivot theory, since an H-complex is an n-dimensional pseudo-manifold. Therefore, viewed properly, path following on V-complexes, which seems to be out of the ordinary, is equivalent to path-following on n-dimensional pseudomanifolds, and can be viewed as the "projection" onto K of path-following on $\bar{K}$.

In most algorithms based on V-complexes, we search for an element of G. We have seen that the set G is derived from the structural properties of ., and hence the way our complex K is divided up into the $A(T)$ is intimately connected to what we can expect to look for in an algorithm on K. Conversely, suppose we wish to find elements x of K with certain labels $L(x)$. G, where G is some set. If we can divide the space into $A(T)$, T . ., such that arises from G, we are close to our stated purpose.

CHAPTER IV

COMBINATORIAL LEMMAS

4.0. Introduction

In Chapter III, we developed the theory of V-complexes and H-complexes, and showed how to trace paths on these complexes. In this chapter, we apply this theory to give constructive proofs of five combinatorial lemmas. We prove Sperner's Lemma [40], and show that Kuhn's algorithm for Sperner's Lemma [44] is a specific instance of path-following on a V-complex. We next prove a generalization of Sperner's lemma. We then prove three lemmas on the n-cube: the Tucker lemma [42], Kuhn's lemma [41], and a new combinatorial lemma on the n-cube.

4.1. Sperner's Lemma and Kuhn's Algorithm

Let $S^n = \{x \in \mathbb{R}^n | x \geq 0, e^T x = 1\}$, and let C be a triangulation of S^n. Note that C induces a triangulation of each face of S^n. Let K^0 consist of the vertices of the triangulation C, and let K be the pseudomanifold corresponding to C. $L(\cdot):K^0 \rightarrow \{1,\ldots,n\}$ is defined to be a <u>proper labelling</u> of K^0 if for each $\mathbf{v} \in K^0$, $v_i = 0$ implies $L(v) \neq i$. A simplex $x \in K$ is said to be <u>completely labelled</u> if $L(x) = \{1,\ldots,n\}$. We can now state and prove:

<u>Lemma 4.1</u> (Sperner's Lemma). Let C triangulate S^n and let $L(\cdot)$ be a proper labelling. Then there are an odd number of completely labelled simplices of C.

PROOF. Our first task is to set up a V-complex on K. To do so, we define $N = \{1,\ldots,n\}$ and $\mathfrak{I} = \{T \subset N \mid n \notin T,\ 1 < i \in T \text{ implies } i-1 \in T\}$. $\mathfrak{I}$ is then the collection:

$$\emptyset,\ \{1\},\ \{1,2\},\ \{1,2,3\},\ \ldots,\ \{1,2,3,\ldots,n-1\}.$$

For $T = \emptyset$, we define $A(T) = \{\emptyset, \{e^1\}\}$. For $\emptyset \neq T \in \mathfrak{I}$, $T = \{1,\ldots,m\}$ for some unique integer m. Then we define $A(T)$ to be the pseudomanifold induced on the face $\{x \in S^n \mid x_i = 0 \text{ for } i > m+1\}$. (This construction is illustrated for $n = 2$ in Figure 4.1.)

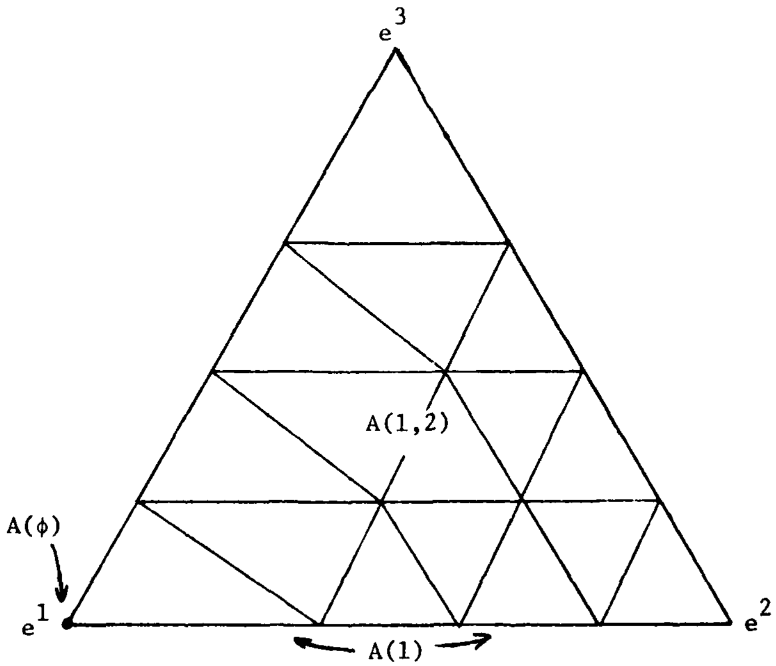


Figure 4.1.

It is simple to verify that $K, \mathfrak{J}$, and $A(\cdot)$ define a V-complex.

Our second task is to examine the set B. We know that $\emptyset \in B$, since $A(\emptyset)$ contains only one 0-simplex. Suppose $\emptyset \neq x = \{v^1,\ldots,v^t\} \in B$. Then $L(x) = T_x$ and $x \in \partial' A(T_x)$. Thus $|x| = |T_x| = t$, and so $T = \{1,\ldots,t\}$. Therefore $L(x) = \{1,\ldots,t\}$. But $x \in \partial' A(T_x)$ implies that there is an i, $1 \leq i \leq t+1$, such that $v_i^j = 0$ for all $j = 1,\ldots,t$. If $i < t+1$, then we cannot have $L(v^j) = i$ for any $j = 1,\ldots,t$ since $L(\cdot)$ is a proper labelling. But $i \in L(x)$. Therefore $i = t+1$. But then $T_x = \{1,\ldots,t-1\}$, a contradiction. Therefore B consists only of $\emptyset$, i.e. $B = \{\emptyset\}$.

Our third task is to examine the set G. Let $x = \{v^1,\ldots,v^{t+1}\}$ be in G.

Then $T_x = \{1,\ldots,t\}$. $L(x) \supset T_x$ and $L(x) \notin \mathfrak{J}$. Therefore either $L(x) = \{1,\ldots,n\}$, or $L(x) = \{1,\ldots,t, s\}$ where $s > t+1$. The latter cannot occur, since for $s > t+1$, $v_s^j = 0$ for all $j = 1,\ldots,t+1$, and hence $s \notin L(x)$ because $L(\cdot)$ is a proper labelling. Thus G is the set of completely labelled simplices.

From Lemma 3.11, we know that G and B have the same parity. Therefore G is odd, proving that there are an odd number of completely labelled simplices. $\emptyset$

Kuhn's algorithm for Sperner's lemma [44] corresponds to following the path from $B = \{\emptyset\}$ to an element of G. Figure 4.2 illustrates this algorithm.

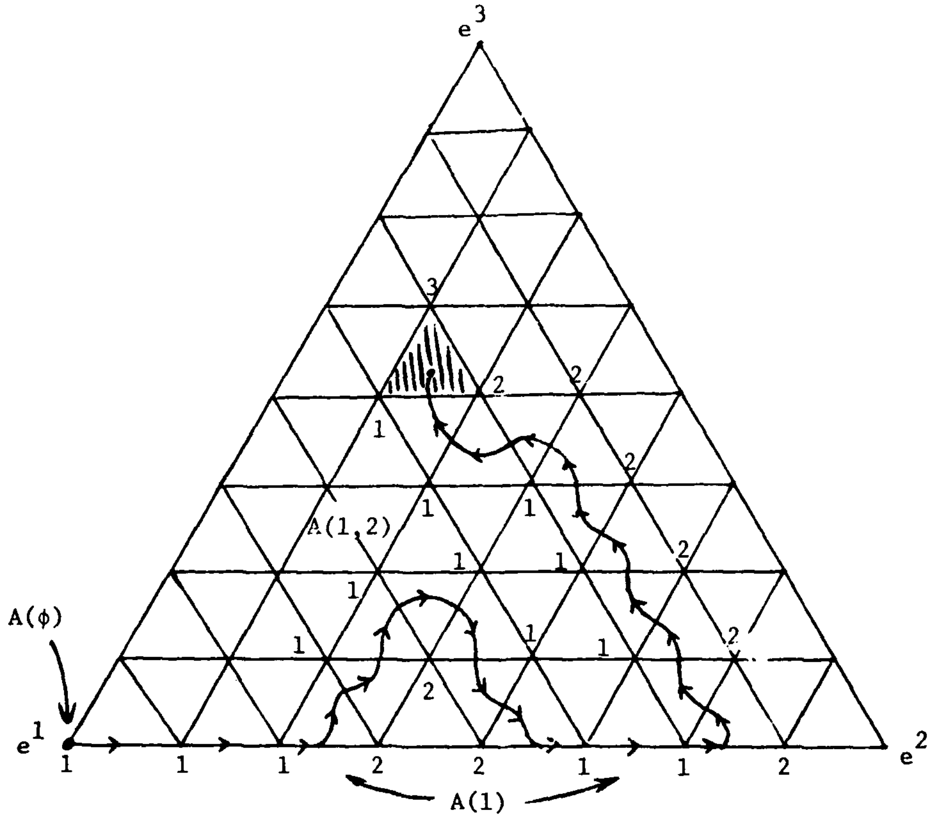


Figure 4.2. Path from ϕ to an element of G.

4.2. A Generalization of Sperner's Lemma

As in Section 4.1, we define $S^n = \{x \in \mathbb{R}^n | x \geq 0, e^T x = 1\}$ and let C be a triangulation of S^n. Let K^0 be the set of vertices of C, and let K be the pseudomanifold corresponding to C. Let $L(\cdot): K^0 \to \{1,\ldots,n\}$ be a labelling function. Let $x = \{v^0,\ldots,v^t\}$ be a simplex of K. We define $S(x) = \{i | v_i^j > 0$ for some $j = 1,\ldots,t\}$. We have the following:

Lemma 4.2. (Generalized Sperner Lemma). Let C triangulate S^n. Then there are an odd number of nonempty simplices x such that $L(x) = S(x)$.

Before proving this lemma, we make a few observations. For a given simplex x, $S(x)$ is the "index set" of the smallest face containing x. Thus if the smallest face of S^n containing x is $\langle e^{i_1},\ldots,e^{i_m}\rangle$, then $S(x) = \{i_1,\ldots,i_m\}$. If $S(x) = L(x)$, then $L(x) = \{i_1,\ldots,i_m\}$. Such a simplex x is said to be "completely labelled in its face." The reason this lemma is a "generalization" of Sperner's lemma is that we have relaxed the requirement that $L(\cdot)$ be a proper labelling; yet we still are able to deduce an interesting conclusion.

PROOF. As in the proof of lemma 4.1, we first construct a V-complex. Let $N = \{1,\ldots,n\}$, and let $\mathfrak{I} = \{T \subset N | n \notin T\}$. We define $A(\emptyset) = \{\emptyset, \{e^n\}\}$, and for $T \in \mathfrak{I}$, $T \neq \emptyset$, we define $A(T)$ to be the pseudomanifold induced by C on the face $\{x \in S^n | x_i = 0, i \neq n, i \notin T\}$. It is simple to verify that K, $A(\cdot)$, and $\mathfrak{I}$ define a V-complex. (This V-complex is illustrated in Figure 4.3, for $n = 2$.)

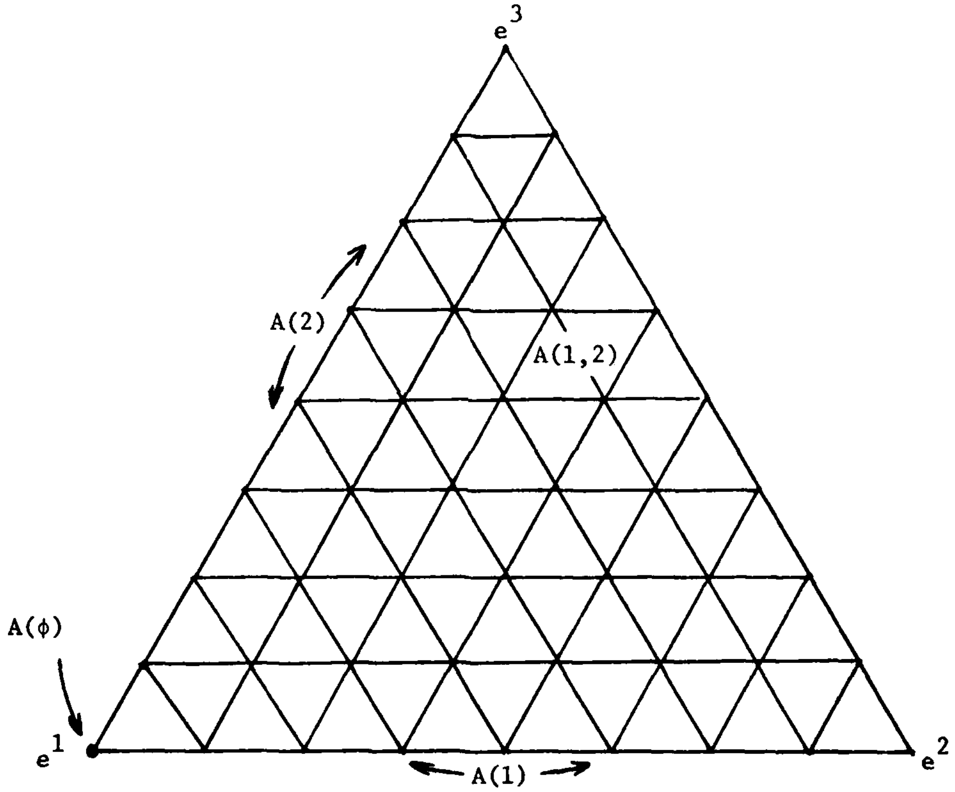


Figure 4.3

Let $x \in G$. Then x is full, and $L(x) \supset T_x$, and $L(x) \notin \mathfrak{S}$. Then $L(x) = T_x \cup \{n\}$. Also, since x is full $S(x) = T_x \cup \{n\}$. Thus x is completely labelled in its face.

Suppose $x \in B$, and $x \neq \emptyset$. Then $L(x) = T_x$, and $x \in \partial' A(T_x)$. Let x be written as $x = \{v^1, \ldots, v^t\}$. $x \in \partial' A(T_x)$ implies $v_n^i = 0$ for all $i = 1, \ldots, t$. Therefore $S(x) = T_x$. Thus x is completely labelled in its face.

Conversely, suppose $S(x) = L(x)$ for a given x. Then if $n \in S(x)$, we must have $L(x) \notin \mathfrak{S}$. But also $L(x) \supset T_x$. Therefore $x \in G$. On the other hand, if $n \notin S(x)$, we have $T_x = S(x) = L(x)$ and $x \in \partial' A(T_x)$. So $x \in B$.

Thus, $G \cup B \setminus \emptyset$ is the set of nonempty x such that $L(x) = S(x)$. By Lemma 3.11, $G \cup B \setminus \{\emptyset\}$ has an odd number of elements, proving the lemma. $\otimes$

An algorithm for computing an element $G \cup B \setminus \{\emptyset\}$ consists of following the path whose endpoint is $\emptyset$. The other endpoint of the path is a simplex $x \neq \emptyset$ for which $L(x) = S(x)$. See Figure 4.4.

As a byproduct of the Generalized Sperner Lemma, we have a lemma due to Scarf [35], which is a "dual" of the Sperner Lemma. In Scarf's lemma, vertices on the boundary receive labels in the complement of the set that gives rise to a proper labelling.

<u>Corollary 4.3</u> (Scarf's Lemma). Let C triangulate S^n, such that no simplex of C has a nonempty intersection with every face of S^n.

Let $L(\cdot):K^0 \to \{1,\ldots,n\}$ be a labelling function such that if $v \in K^0$, $v \in \partial S^n$, $v_i > 0$ then $L(v) \neq i$. Then there are an odd number of simplices x such that $L(x) = \{1,\ldots,n\}$.

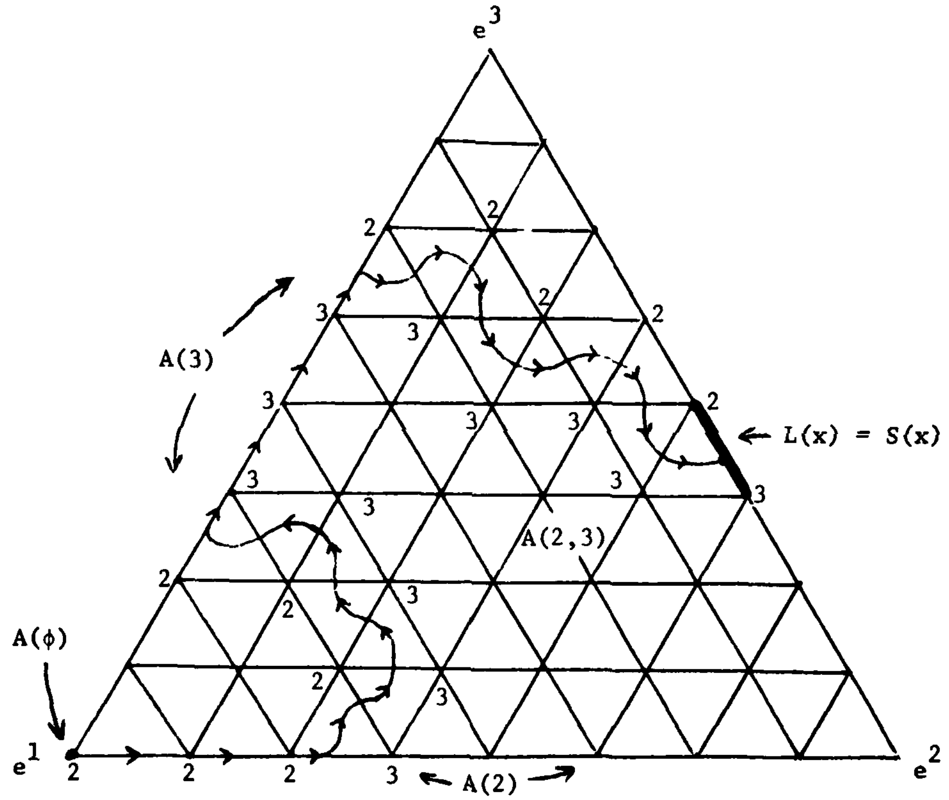


Figure 4.4. Path from ϕ to an element x such that $L(x) = S(x)$.

PROOF. The proof follows directly from the Generalized Sperner Lemma. Since $L(\cdot):K^0 \to \{1,\ldots,n\}$, there are an odd number of simplices x that are completely labelled in their face. Let x be one such simplex. Suppose $L(x) = T \neq \{1,\ldots,n\}$. Then for each $v \in x$, $v_i = 0$ for $i \notin T$. But by design of $L(\cdot)$, for each $i \in T$, there is a $v \in x$ such that $v_i = 0$. Thus x meets every face of S^n, a contradiction. Thus $L(x) = \{1,\ldots,n\}$. ⊗

4.3. Tucker's Lemma on the *n*-Cube

Let p be a positive integer and let $C^n = \{x \in \mathbb{R}^n \mid -pe \leq x \leq pe\}$, an n-dimensional cube in $\mathbb{R}^n$. Let C be Todd's Union-Jack triangulation [41] of $\mathbb{R}^n$ restricted to C^n. Let K^0 be the vertices of C and let K be the pseudomanifold corresponding to C. Let $x = \{v^0,\ldots,v^t\} \in K$. Then, by the symmetry of the Union-Jack triangulation, $\{-v^0,\ldots,-v^t\} \in K$. We define $-x \triangleq \{-v^0,\ldots,-v^t\}$. Let $L(\cdot):K^0 \to \{1,\ldots,n,-1,\ldots,-n\}$ be a labelling of K^0 such that $v \in \partial K$ implies $L(v) = -L(-v)$, i.e. $L(\cdot)$ is odd on the boundary K.

Note that K^0 is simply the integral points of C^n.

We have the following:

Lemma 4.4 (Tucker's Lemma). Let $L(\cdot)$ be a labelling of the integral points of C^n, which is odd on the boundary of C^n. Then there exists integral points v', v'', such that $\|v'-v''\|_\infty = 1$, and $L(v') = -L(v'')$.

PROOF. We first construct a V-complex. Let $N = \{1,\ldots,n,-1,\ldots,-n\}$, and let $\mathfrak{I} = \{T \subset N \mid i \in T$ implies $-i \notin T\}$. Let $A(\emptyset) = \{\emptyset, \{0\}\}$, and for $\emptyset \neq T \in \mathfrak{I}$, let $A(T)$ be the pseudomanifold corresponding to the restriction of C to the region

$$\{x \in \mathbb{R}^n \mid i x_{|i|} \geq 0 \text{ for } i \in T, \text{ and } x_i = 0 \text{ if neither } i \text{ nor } -i \in T\}.$$

It is simple to verify that K, $\mathfrak{I}$, and $A(\cdot)$ define a V-complex. (Such a V-complex is illustrated in Figure 4.5, for $n = 2$ and $p = 4$.)

Let us now examine the set B. $\emptyset \in B$, so B has at least one element. Let $\emptyset \neq x \in B$. Then $x \in \partial' A(T_x)$. For any $\emptyset \neq T \in \mathfrak{I}$, $\partial' A(T) \subset \partial K$.

Therefore, $x \in \partial K$. Also $L(x) = T_x$. Furthermore, $-x \in \partial K$, and $L(-x) = \{-i \mid i \in T_x\}$, and in fact $T_{-x} = \{-i \mid i \in T_x\}$. Therefore, $-x \in B$. Thus we see that except for $\emptyset$, B consists of pairs of the form $x, -x$. Therefore B has an odd number of elements, and so must G, by Lemma 3.11.

Thus there is an element $x \in G$. Thus there are two vertices of x, say v' and v'', such that $L(v') = -L(v'')$. And since v' and v'' are elements of x, $\| v' - v'' \|_\infty = 1$, proving the lemma. ⊗

An algorithm for finding a pair v', v'' consists of following the path that orignates with $\emptyset$. If its endpoint is an element of G, stop. If it is an element x of B, reinitiate the path at $-x$. Continuing in this fashion, an element of G will be found. For a complete description of the pivot rules for this algorithm, see Freund and Todd [11]. See Figure 4.6 for a sample path.

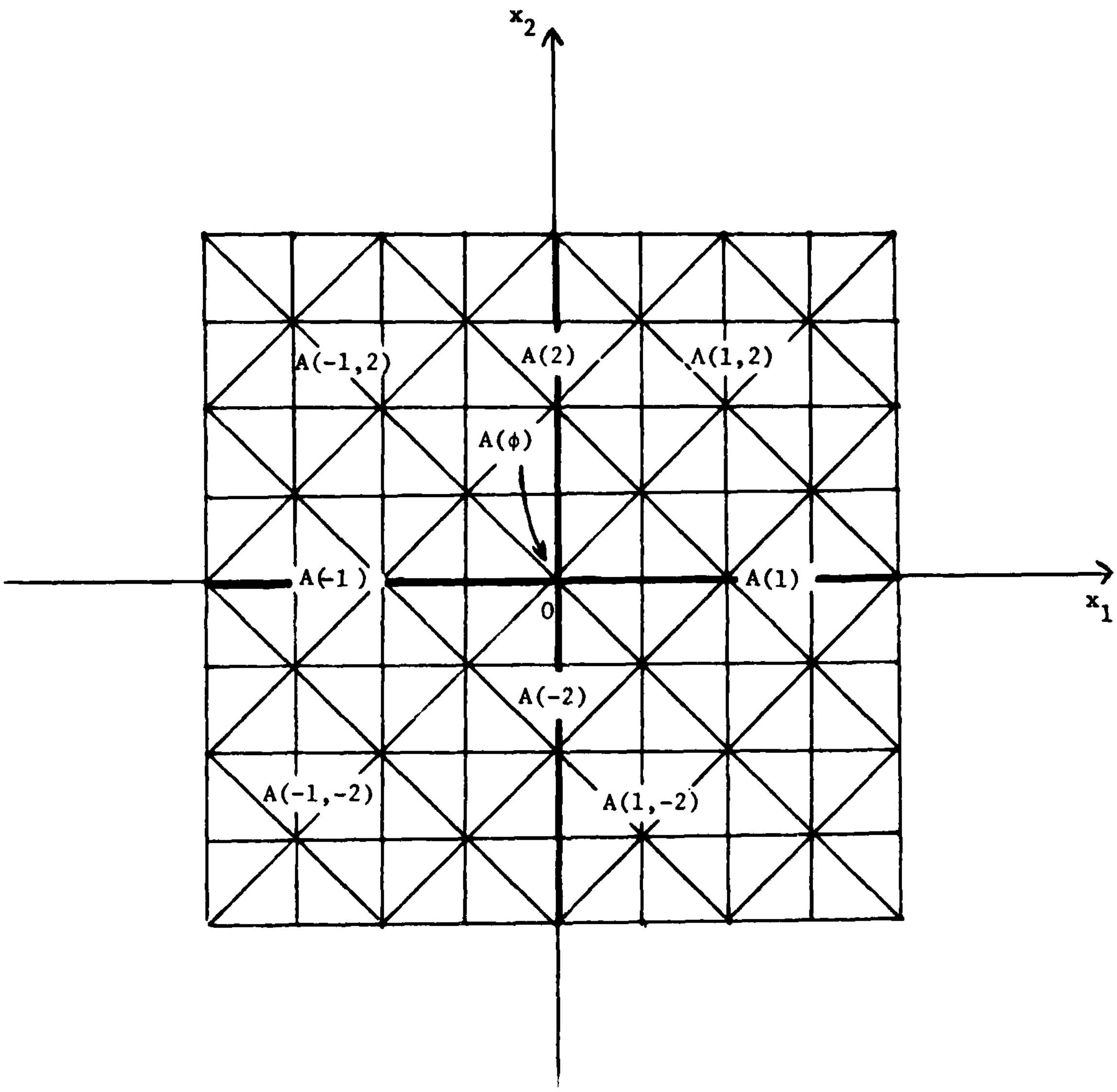


Figure 4.5

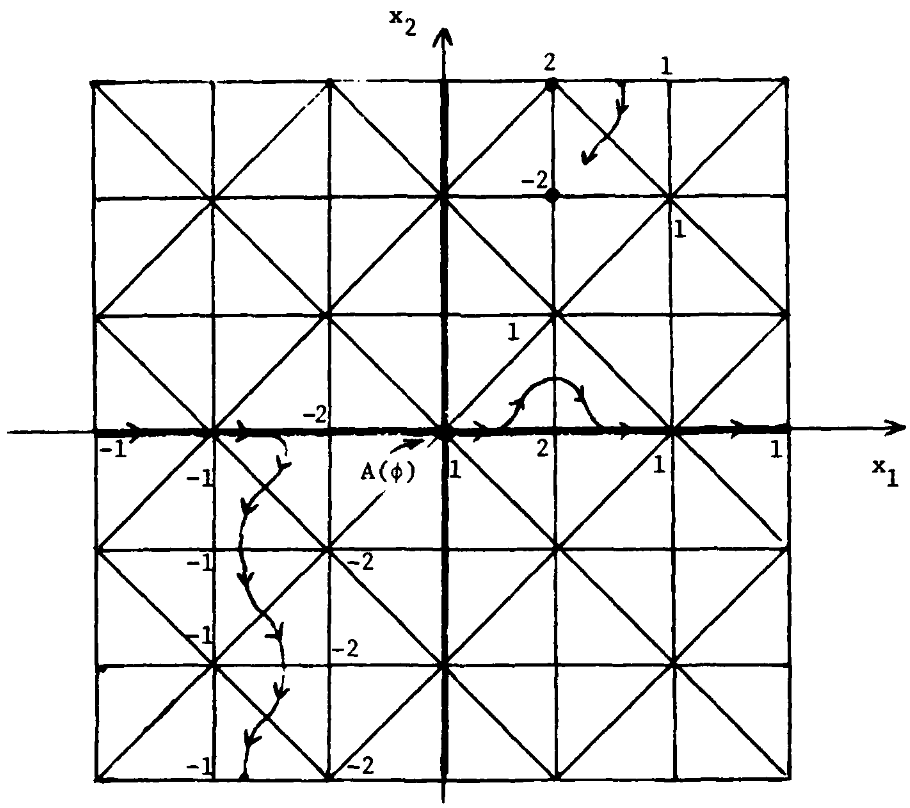


Figure 4.6.

Unlike the first two combinatorial lemmas of this chapter, we cannot assert that there are an odd number of pairs v', v'' with $L(v') = -L(v'')$ and $\| v'-v'' \|_\infty = 1$. This is because not all such pairs are subsets of elements of G. Nevertheless, we can assert that there is at least one pair. Figure 4.7 illustrates an instance of Tucker's lemma where the number of such pairs is six, an even number.

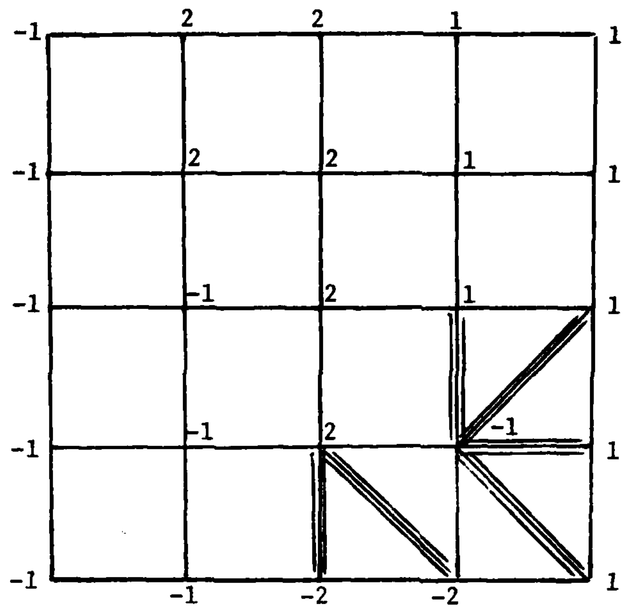


Figure 4.7.

4.4. <u>Kuhn's Lemma</u>

Let p be a positive integer and let $C^n = \{x \in \mathbb{R}^n \mid 0 \leq x \leq pe\}$, an n-dimensional cube in $\mathbb{R}^n$. Let C be Kuhn's triangulation of $\mathbb{R}^n$ restricted to C^n. Let K^0 be the vertices of C, i.e. the integral points of C^n. Let K be the pseudomanifold corresponding to C. Let $I = \{y \in \mathbb{R}^n \mid y_i = 0 \text{ or } 1 \text{ for } i = 1,\dots,n\}$. Let $\ell(\cdot): K^0 \to I$ be a function such that

i) $v_i = 0$ implies $\ell_i(v) = 0$

ii) $v_i = p$ implies $\ell_i(v) = 1$.

We define $L(v)$ = the number of leading zeroes of $\ell(v)$ for each $v \in K^0$. We have the following:

Lemma 4.5 (Kuhn's lemma). There exists an odd number of simplices $x \in K$ such that $L(x) = \{0,1,\ldots,n\}$.

PROOF. We first construct a V-complex. Let $N = \{0,1,\ldots,n\}$, and let $\mathfrak{I} = \{T \subset \{1,\ldots,n\} \mid 0 < i \in T \text{ implies } i-1 \in T\}$. $\mathfrak{I}$ then is the collection

$$\emptyset,\ \{0\},\ \{0,1\},\ \{0,1,2\},\ \ldots,\ \{0,1,\ldots,n-1\}\ .$$

We define $A(\emptyset) = \{\emptyset,\{pe\}\}$, and for $\emptyset \neq T \in \mathfrak{I}$, $T = \{0,\ldots,m\}$ for some $m < n$. We then define $A(T)$ to be the pseudomanifold corresponding to the restriction of C to

$$\{x \in C^n \mid x_i = p \quad \text{for} \quad i > m+1\}\ .$$

It is simple to verify that K, $\mathfrak{I}$, and $A(\cdot)$ define a V-complex. (Figure 4.8 illustrates such a V-complex for $n = 2$.)

Let us now examine the set B. We know that $\emptyset \in B$. Suppose $\emptyset \neq x \in B$, where $x = \{v^1,\ldots,v^m\}$ for some m. Then $T_x = \{0,\ldots,m-1\}$ and $L(x) = T_x$ and $x \in \partial' A(T_x)$. Since $x \in \partial' A(T_x)$, either $v_i^j = 0$ for all $j = 1,\ldots,m$ and some $i \in \{1,\ldots,m\}$, or $v_i^j = p$ for all $j = 1,\ldots,m$ and some $i \in \{1,\ldots,m\}$. Suppose the former is true. Then $i-1 \notin L(x)$, a contradiction. If the latter is true, then $i \notin L(x)$, which is a contradiction unless $i = m$. But then $x \in A(\{0,\ldots,m-2\})$, so that $T_x = \{0,\ldots,m-2\}$, a contradiction. Therefore $B = \{\emptyset\}$.

Next we examine the set G. Let $x \in G$. Then $T_x = \{0,\ldots,m\}$ for some $m < n$, and $L(x) \supset T_x$, $L(x) \notin \mathfrak{I}$. Therefore either $L(x) = \{0,1,\ldots,n\}$, or $m < n-1$ and there is an $s > m+1$ such that

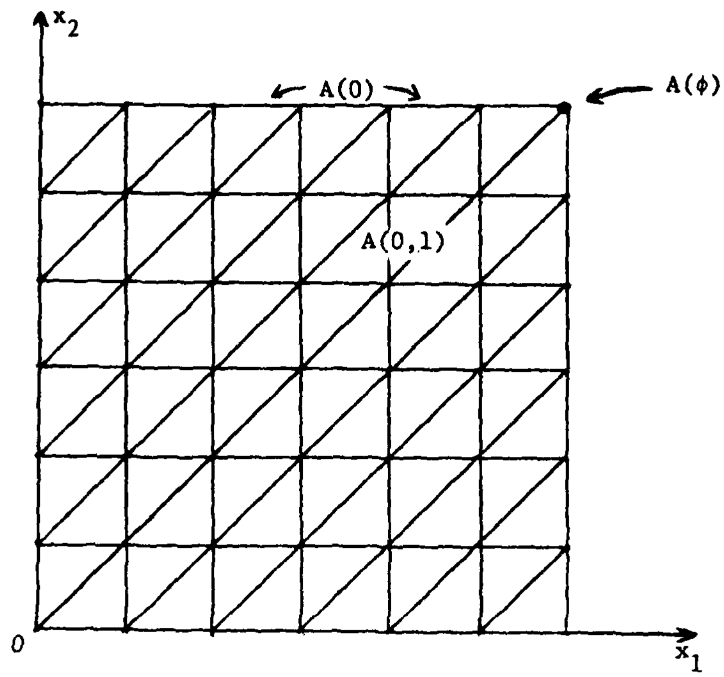


Figure 4.8

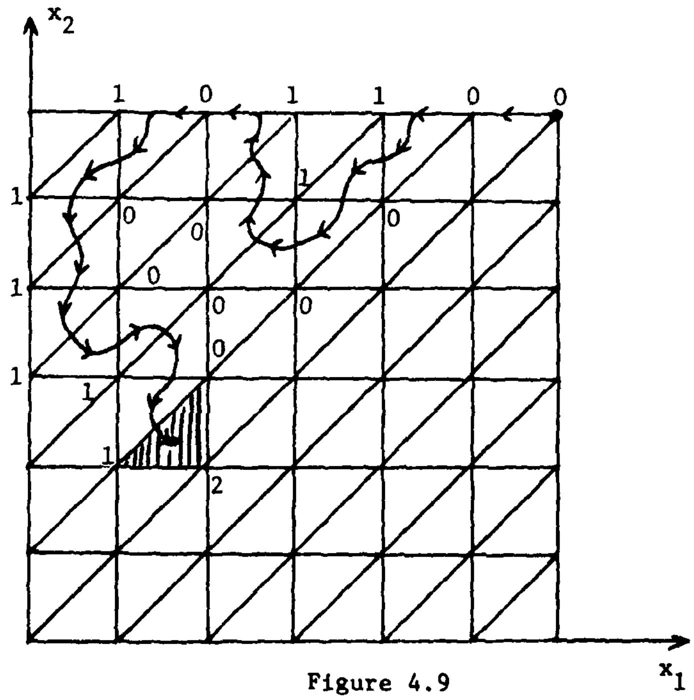


Figure 4.9

$L(x) = \{0,\ldots,m,s\}$. Suppose the latter is true. Let $x = \{v^0,\ldots,v^{m+1}\}$. Since $T_x = \{0,\ldots,m\}$, $v^j_{m+2} = p$ for all $j \in \{0,\ldots, m+1\}$. But then $L(v^j) \leq m+1$ for all j, and so $s \notin L(x)$, a contradiction. Therefore $L(x) = \{0,\ldots,n\}$.

Furthermore, if $L(x) = \{0,\ldots,n\}$, then clearly $x \in G$. Therefore G consists precisely of those x for which $L(x) = \{0,\ldots,n\}$. By Lemma 3.11, G has an odd number of elements, which proves the lemma. ⊗

An algorithm for finding an element of G consists of following the path starting at $\emptyset \in B$, and terminating at its other endpoint, an element of G. See Figure 4.9 for a sample path.

4.5. Another Combinatorial Lemma on the n-Cube

Let p be a positive integer and let $C^n = \{x \in \mathbb{R}^n | 0 \leq x \leq pe\}$. Let C be a triangulation of C^n. Let K^0 be the vertices of C, and let K be the pseudomanifold corresponding to C. Let $L(\cdot)$: $K^0 \to \{1,\ldots,n,-1,\ldots,-n\}$ be a labelling of K^0, with the restriction on $L(\cdot)$ that

i) $v \in K^0$, $v_i = 0$ implies $L(v_i) \neq -i$, and

ii) $v \in K^0$, $v_i = p$ implies $L(v_i) \neq i$.

We have the following:

<u>Lemma 4.6</u>. Let C triangulate C^n and let $L(\cdot)$ be as above. Then there exists a pair of vertices v', v'' in some simplex of C such that $L(v') = -L(v'')$.

PROOF. Let $N = \{1,\ldots,n,-1,\ldots,-n\}$ and let $\mathfrak{J} = \{T \subset N | i \in T$ implies $i \geq 0\}$. Let $A(\emptyset) = \{\emptyset,\{0\}\}$, and for $\emptyset \neq T \in \mathfrak{J}$, we define $A(T)$ to be the pseudomanifold corresponding to the restriction of C to

$$\{x \in C^n | i \notin T \text{ implies } x_i = 0\} .$$

It is simple to verify that K, $\mathfrak{J}$, and $A(\cdot)$ define a V-complex. Figure 4.10 illustrates such a V-complex for $n = 2$.

First let us examine the set B. We know that $\emptyset \in B$, since $A(\emptyset)$ has only one 0-simplex. Let $\emptyset \neq x = \{v^1,\ldots,v^t\} \in B$. Then $L(x) = T_x$ and $x \in \partial' A(T_x)$. Thus for some $i \in T_x, v_i^j = p$ for all $j = 1,\ldots,t$. But then $L(v_i^j) \neq i$ for any $j = 1,\ldots,t$. Thus $i \notin L(x)$, a contradiction. Therefore $B = \{\emptyset\}$.

By Lemma 3.11, G must have an odd number of elements, and hence at least one. Let $x \in G$. Then $L(x) \supset T_x$ and $L(x) \in \mathfrak{J}$. This means there is an $i > 0$ such that $L(x) = -i$. Suppose $i \notin T_x$. Then for each $v \in x$, $v_i = 0$ and so by the restriction on $L(\cdot)$, $-i \notin L(x)$, a contradiction. Therefore $\{i,-i\} \in L(x)$. Thus there are two elements v' and v'' of x such that $L(v') = -L(v'')$, proving the lemma. ⊗

An algorithm for finding v', v'' consists of following the path whose endpoint is $\emptyset \in B$, until an element of G is found. Figure 4.11 illustrates a sample path. As in the case of the Tucker lemma, we cannot assert that there are odd number of pairs, since not all pairs are contained in elements of G.

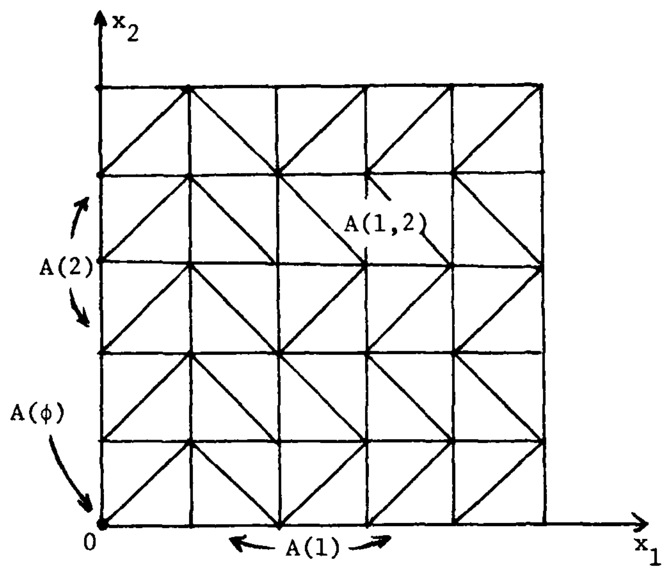


Figure 4.10

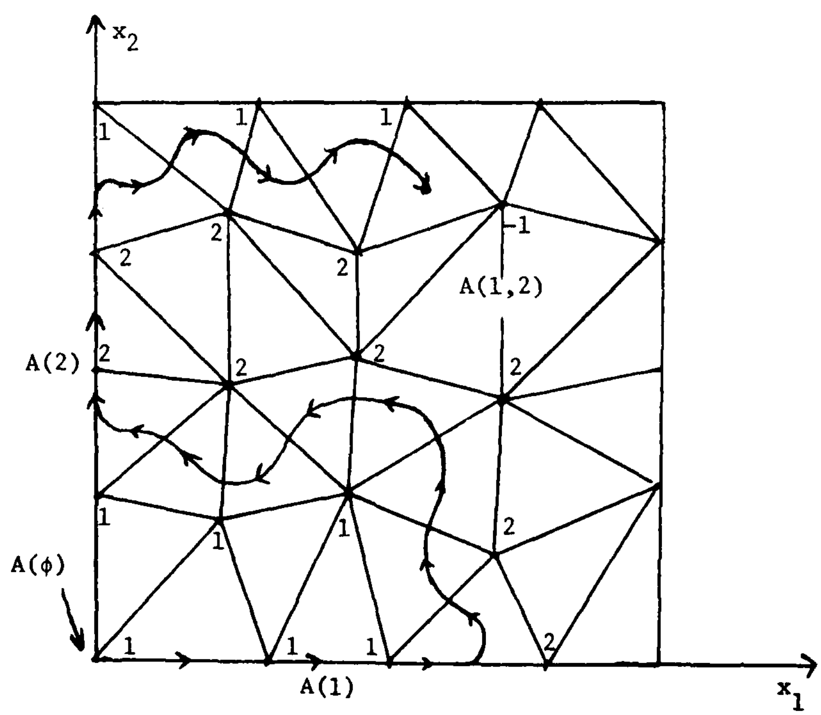


Figure 4.11

4.6. Concluding Remarks

In this chapter, we have given "constructive" proofs of six combinatorial lemmas in topology. The proofs are constructive in the sense that, for most standard triangulations of the n-simplex and the n-cube, it is possible to write down the pivot rules of an algorithm that will compute the simplices of a path from $\emptyset \in B$ to the desired simplex as stated in the conclusion of the lemma.

Sperner's lemma leads to a very elegant proof the Brouwer fixed-point theorem. In the next chapter, we show that Kuhn's lemma and Lemma 4.6 also lead (independently) to a proof of this famous fixed-point theorem. Tucker's lemma can be used to provide a proof of the Borsuk-Ulam and Lusternik-Schnirelmann antipodal point theorems [11,27].

The Generalized Sperner Lemma (Lemma 4.2) has been independently developed by Ky Fan [9], and inadvertently by Lüthi [31]. The algorithm of Lüthi in [31] for the nonlinear complementarity problem is precisely the path following routine used in our proof the Generalized Sperner Lemma.

In his 1960 paper, "Some Combinatorial Lemmas in Topology" [21], Kuhn derives the Tucker lemma from his lemma, for the case $n = 2$. He asks "Is there a derivation of Tucker's Lemma from the Strong Cubical Sperner [Kuhn] Lemma for all n?" [21]. Although this question is still unresolved, we see that both Tucker's and Kuhn's lemma are specific instances of labelling on V-complexes, and that the proofs of both are similar in nature.

CHAPTER V

EXTENSIONS OF THE COMBINATORIAL LEMMAS

5.0. Introduction

In Chapter IV, we demonstrated six combinatorial lemmas. In this chapter, we present various extensions of these lemmas. In Section 5.1, we show the relationship between Brouwer's fixed-point theorem and five of the combinatorial lemmas. In Section 5.2, we use the Generalized Sperner Lemma to prove other mathematical results. In Section 5.3, we present a "homotopy"-type V-complex and suggest an algorithm based on the Generalized Sperner Lemma. In Section 5.4, we show the relationship between the Tucker lemma and antipodal point theorems.

5.1. Brouwer's Theorem and Combinatorial Lemmas

Simplicial methods were first developed in the 1960's for the computation of fixed points of continuous mappings, the existence of which was first demonstrated by Brouwer's celebrated Fixed-Point Theorem. It is only appropriate therefore to show the relationship between Brouwer's Theorem and five of the combinatorial lemmas of Chapter IV. Although some of the material of this section is not new, it is included for the sake of completeness. Brouwer's Theorem can be stated as follows:

Theorem 5.1 (Brouwer). Let $f(\cdot):S \to S$, where S is a compact convex set in $\mathbb{R}^n$, and $f(\cdot)$ is continuous. Then there exists a fixed point of $f(\cdot)$, i.e. a point $x^* \in S$ such that $f(x^*) = x^*$.

An elegant proof of Brouwer's Theorem is provided by using Sperner's Lemma. We have

Lemma 5.2. Sperner's Lemma implies Brouwer's Theorem.

PROOF: Without loss of generality, we can assume $S = S^n \triangleq \{x \in \mathbb{R}^n | e^T x = 1, x \geq 0\}$. Let C be a triangulation of S and consider the following labelling function on the vertices of C:

$$L(v) = i \quad \text{if} \quad v_i > 0 \quad \text{and} \quad f_i(v) \leq v_i \ .$$

If more than one such i exists for a particular vertex v, let $L(v)$ be the smallest such i. $L(\cdot)$ is readily seen to be a proper labelling of the vertex set of C, and so by Sperner's lemma, there exists a completely labelled simplex. If we take an infinite sequence of triangulations C with diameter approaching zero, the sequence of completely labelled simplices must have at least one subsequence that converges to a single point, say x^*. At this point, we have $f_i(x^*) \leq x_i^*$, $i = 1,\ldots,n$, since $f(\cdot)$ is continuous. But since $e^T \cdot f(x^*) = e^T x^* = 1$, this means that $f(x^*) = x^*$, proving the theorem. ⊗

Furthermore, we have:

Lemma 5.3. Brouwer's Theorem implies Sperner's Lemma.

PROOF: Let C triangulate S^n and let $L(\cdot)$ be a proper labelling of K^0, the vertices of C. For each $v \in K^0$, define $f(v) = e^{L(v)-1}$, and for the purposes of this proof alone, we define $e^0 = e^n$. Extend $f(\cdot)$ to all of S^n in a piece-wise linear manner on each subsimplex. Then $f(\cdot)$ is continuous and maps S^n into S^n. Thus there is a fixed point x^* of f. Let $v^1,\ldots,v^m$ be the unique smallest simplex of C that contains x^*. We have $x^* = \Sigma\, \bar{\lambda}_i v^i = \Sigma\, \bar{\lambda}_i\, f(v^i)$ for appropriate $\bar{\lambda}$, where $e^T\bar{\lambda} = 1$, and $\bar{\lambda} \geq 0$. Suppose $x^* \not> 0$. Then there is an i such that $x^*_i = 0$. But then $v^j_i = 0$ for all j, whereby $i \notin \{L(v^1), \ldots, L(v^m)\}$, since $L(\cdot)$ is a proper labelling. This implies that $f_{i-1}(x^*) = 0$, and hence $x^*_{i-1} = 0$. (If $i = 1$, it implies $x^*_n = 0$.) Proceeding likewise, we have that $x^* = 0$, a contradiction. Therefore $x^* > 0$, and since $f(x^*) = \Sigma\, \bar{\lambda}_i \cdot e^{[L(v^i)-1]}$, we must have that $\{L(v^1), \ldots, L(v^m)\} = \{1,\ldots,n\}$, and so $m = n$. Therefore, the simplex $\{v^1, \ldots, v^n\}$ is completely labelled, completing the proof. ⊗

Actually, we have not proved Sperner's Lemma in its entirety, since we have not shown that there are an odd number of completely labelled simplices, so the equivalence between Brouwer's Theorem and Sperner's Lemma is not complete.

Similarly, we can show:

Lemma 5.4. Brouwer's Theorem implies the Generalized Sperner Lemma.

PROOF: Let C triangulate S^n and let $L(\cdot)$ be a labelling function as in the Generalized Sperner Lemma. For each v in the vertex set K^0 of C, define $f(v) = e^{L(v)}$, and extend this map in a piece-wise linear manner. f satisfies the conditions of Brouwer's Theorem, and so there exists a fixed point of f, say x^*. Let $\langle v^1, \ldots, v^m\rangle$ be the unique smallest simplex of C that contains x^*. We have $x_i^* > 0$ if and only if $i \in S(\{v^1,\ldots, v^n\})$, where $S(\cdot)$ is defined in Section 4.2. Furthermore, by the construction of $f(\cdot)$, $x_i^* > 0$ if and only if $i \in \{L(v^1), \ldots, L(v^m)\}$. Thus there exists a simplex of C that is completely labelled in its face, proving the lemma. ⊗

As in Lemma 5.3, we have not proven the Generalized Sperner in its full force, since we have not shown that there are an odd number of simplices that are completely labelled in their faces. Finally, we show:

Lemma 5.5. The Generalized Sperner Lemma implies Sperner's Lemma.

PROOF: Let $S^n = \{x \in \mathbb{R}^n \mid x \geq 0,\ e^Tx = 1\}$. Our proof is by induction on n. For $n = 1$, the two lemmas are trivially identical. Suppose the implication is true for all $k < n$. Let $N = \{1,\ldots,n\}$ and for $\emptyset \neq T \subset N$, define $S^T = \{x \in \mathbb{R}^n \mid x \geq 0,\ e^Tx = 1,\ x_i = 0 \text{ for } i \notin T\}$. Let C triangulate $S^n = S^N$ with vertex set K^0. Let $L(\cdot)$ be a proper labelling of S^n, and note $L(\cdot)$ is a "proper" labelling of S^T, i.e. $v \in K^0 \cap S^T$ implies $L(v) \in T$. Thus for $\emptyset \neq T$, $N \neq T$, we inductively have that there are an odd number of simplices in S^T that have

label set T. We have

$$\begin{aligned} &\#\text{ simplices in } S^n \text{ that are completely labelled} \\ &\quad + \sum_{\substack{\emptyset \neq T \neq N \\ T \subset N}} (\#\text{ simplices in } S^T \text{ that have label set } T) \\ &\quad = \text{an odd number,} \end{aligned}$$

by the Generalized Sperner lemma. Each term in our summation Σ is odd, and there are $(2^n - 2)$ terms, an even number for $n > 1$. Thus the total number in the summation term is even. Hence, the number of simplices in S^n that are completely labelled is odd, proving Sperner's lemma. ⊗

Our next task is to show the relationship between Brouwer's Theorem and Kuhn's Lemma. We have:

<u>Lemma 5.6</u>. Kuhn's Lemma implies Brouwer's Theorem.

<u>PROOF:</u> Here let $S = C^n = \{x \in \mathbb{R}^n \,|\, 0 \leq x \leq e\}$, and let $f(\cdot): S \to S$ be continuous. Let C be a triangulation of C^n with vertex set K^0. For each vertex $v \in K^0$, let

$$\ell_i(v) = \begin{cases} 0 & \text{if } f_i(v) \geq x_i \neq 1 \\ 1 & \text{if } f_i(v) \leq x_i \neq 0 \,, \end{cases} \qquad i = 1, \ldots, n.$$

If there is more than one choice for $\ell_i(v)$, choose $\ell_i(v) = 0$. Let $L(v)$ be as described in Section 4.4. Note $\ell(\cdot)$ satisfies the conditions of Kuhn's Lemma. Thus there exists a simplex of C that has labels

$\{0,1,\ldots,n\}$. Take a sequence of such simplices as the diameter of C approaches 0; such a sequence will have a cluster point, say x^*. By the continuity of $f(\cdot)$, we have $f_i(x^*) \leq x_i^*$ and $f_i(x^*) \geq x_i^*$ for $i = 1,\ldots,n$. Thus $f(x^*) = x^*$, proving Brouwer's Theorem. ⊗

We cannot assert that Brouwer's Theorem implies Kuhn's Lemma. However, there is a weak form of Kuhn's lemma, called the Cubical Sperner Lemma in [21], that as Kuhn shows, implies Brouwer's Theorem. See [21] for details of this proof.

Last of all, we show the relationship between Lemma 4.6 and Brouwer's Theorem. We have:

Lemma 5.7. Lemma 4.6 implies Brouwer's Theorem.

PROOF: Let $S = C^n = \{x \in \mathbb{R}^n \mid 0 \leq x \leq e\}$, and let $f(\cdot): S \to S$ be continuous. Let C be a triangulation of C^n with vertex set K^0. We define for each $v \in K^0$,

$$L(v) = \begin{cases} i & \text{if } \|f(v) - v\|_\infty = f_i(v) - v_i, \quad v_i \neq 1 \\ -i & \text{if } \|f(v) - v\|_\infty = v_i - f_i(v), \quad v_i \neq 0 \end{cases}$$

If there is more than one choice for i, let i be the smallest such index. $L(\cdot)$ satisfies restrictions (i) and (ii) of Section 4.5, and so by Lemma 4.6, there exists a pair of vertices v' and v'' of K^0 in some simplex of C, such that $L(v') = -L(v'')$. As we let the diameter

of C go to zero, and take a limiting sequence of the pairs (v',v''), we must have at least one cluster point, say x^*. But by the continuity of $f(\cdot)$, we have $\|f(x^*) - x^*\|_\infty = 0$, and so x^* is a fixed-point, proving Brouwer's Theorem. ⊗

Finally, we remark that Scarf's Lemma (Corollary 4.3) is equivalent to Brouwer's Theorem. The proof follows along the same lines as Lemmas 5.2 and 5.3.

By way of concluding this section, Figure 5.1 shows the relationships contained herein, where "→" denotes "implies" in the direction of the arrow.

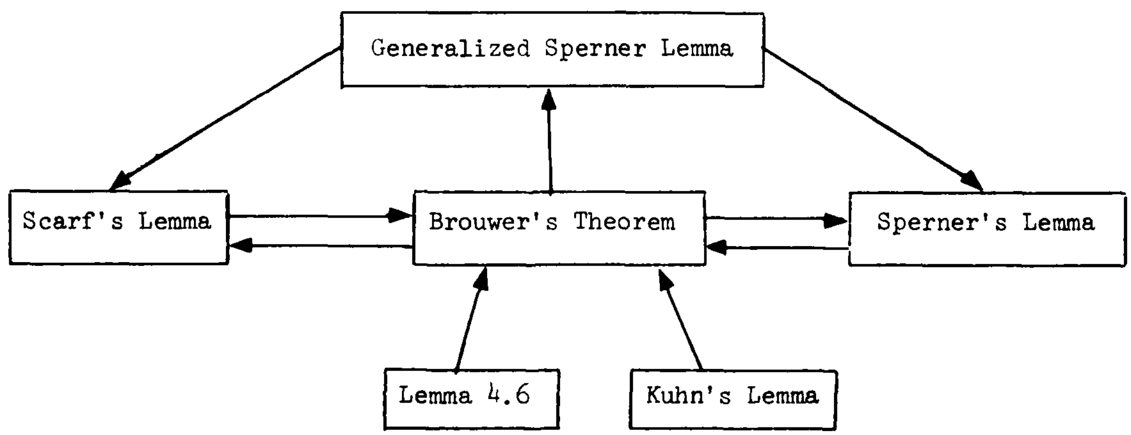


Figure 5.1

5.2. Extensions of the Generalized Sperner Lemma

In this section we prove four mathematical results that are by-products of the Generalized Sperner Lemma.
The first result, a covering theorem, is used to prove the other three results. Throughout this section, let $S^n = \{x \in \mathbb{R}^n | e^T x = 1, x \geq 0\}$. We have

Theorem 5.9 (Covering Theorem). Let C^i, $i = 1,\ldots,n$ be closed sets in $\mathbb{R}^n$ such that $\cup_{i=1}^n C^i \supset S^n$. Then there is at least one point x^* in S^n such that $\{i | x_i^* > 0\} \subset \{i | x^* \in C^i\}$.

PROOF: Let C triangulate S^n with vertex set K^0, and let $L(\cdot)$ be a labelling of K^0, where for each $v \in K^0$,

$$L(v) \in \{i | v \in C^i\} .$$

By Lemma 4.2, there is a simplex x of C that is completely labelled in its face. Take a sequence of triangulations whose diameter goes to zero in the limit. Then there is a sequence of simplices x, completely labelled in their faces, that have a limit point, say x^*. Since each C^i is a closed set, we have $\{i | x_i^* > 0\} \subset \{i | x^* \in C\}$, proving the theorem. $\otimes$

This theorem is illustrated in Figures 5.2 and 5.3. In Figure 5.3, a type of "degeneracy" occurs at x^*, showing that strict inclusion "$\subsetneq$" of the theorem can indeed occur.

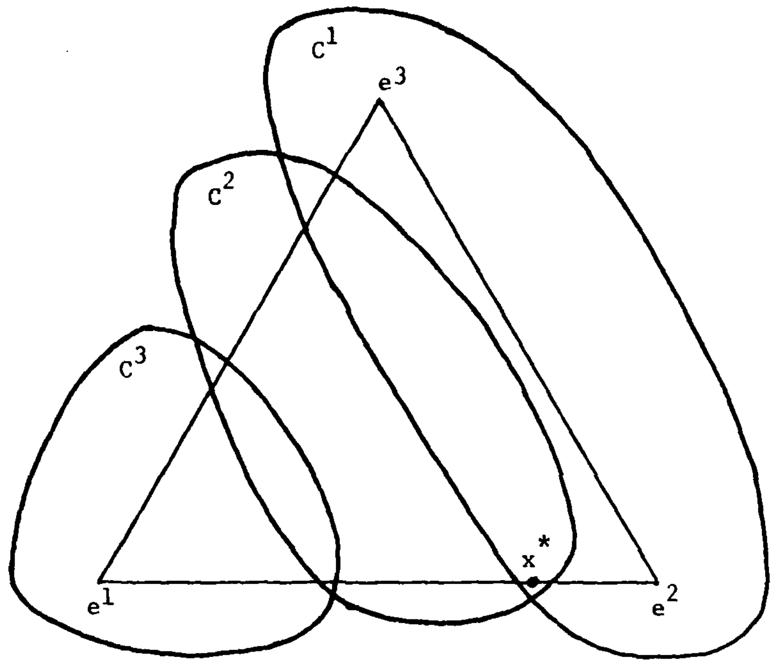


Figure 5.2

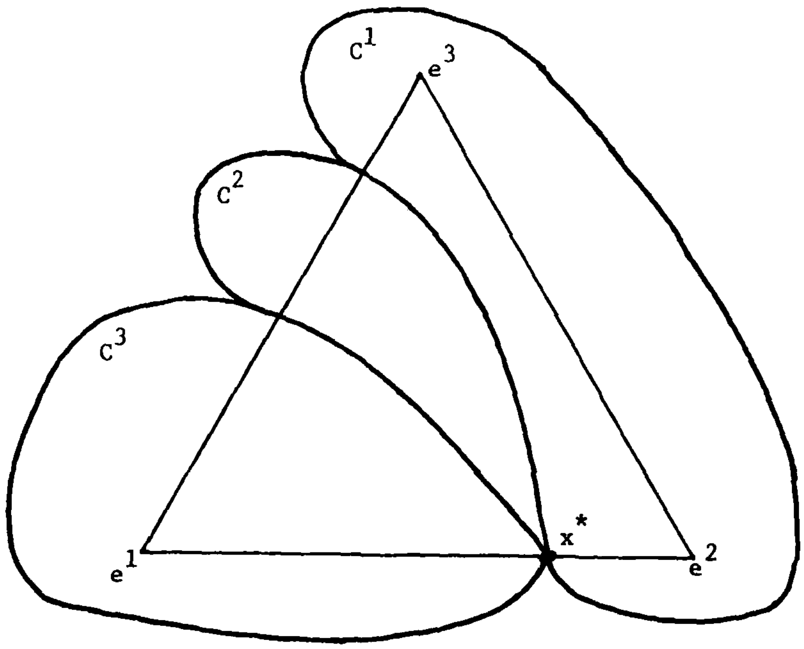


Figure 5.3

Theorem 5.9 can be generalized. Let S be an n-simplex in $\mathbb{R}^n$ and for each $x \in S$, let $\alpha(x)$ be the barycentric representation of x. Then we have

<u>Theorem 5.10</u> (Covering Theorem). Let C^i, $i = 0,\ldots,n$, be closed sets in $\mathbb{R}^n$ such that $\bigcup_{i=0}^{n} C^i \supset S$. Then there is at least one point x^* in S such that

$$\{i \mid \alpha_i(x^*) > 0\} \subset \{i \mid x^* \in C^i\}. \qquad \otimes$$

Our next lemma demonstrates the existence of stationary points (see Eaves [3], and Lüthi [31]). Let $S^{n+1} = \{(x,w) \in \mathbb{R}^{n+1} \mid e^T x + w = 1, x \geq 0,\ w \geq 0\}$, where it is understood that $x \in \mathbb{R}^n$. Let $D^n = \{x \in \mathbb{R}^n \mid e^T x \leq 1,\ x \geq 0\}$. Clearly D^n is the projection of S^{n+1} onto the x-coordinates. Let $f(\cdot): D^n \to \mathbb{R}^n$ be continuous. A point x in D^n is said to be a <u>stationary point</u> of the pair (f, D^n) (see Eaves [3]) if and only if there exists $z \in \mathbb{R}$ and $y \in \mathbb{R}^n$ such that

i) $y \geq 0,\ z \geq 0$

ii) $f(x) = y - ze$

iii) $x \cdot y = 0$

iv) $z(1 - e^T x) = 0$.

We have the following:

Lemma 5.11 (Hartman and Stampacchia [15], and Karamardian [17a], [17b], [18]). There exists a stationary point x^* of (f,D^n).

PROOF: Our proof is based on Theorem 5.9. For $i = 1,\dots,n$, define

$$C^i = \{(x,w) \in S^{n+1} \mid f(x) \not\geq 0 \text{ and } f_i(x) \leq f_j(x) \text{ for any } j = 1,\dots,n\}$$

Define $C^{n+1} = \{(x,w) \in S^{n+1} | f(x) \geq 0\}$. Note that each C^i $(i = 1,\dots,n+1)$ is closed, and that $\cup_{i=1}^{n+1} C^i \supset S^{n+1}$. Thus by Theorem 5.9, there exists (x^*,w^*) in S^{n+1} such that

i) $x_i^* > 0$ implies $(x^*,w^*) \in C^i$, $i = 1,\dots,n$, and

ii) $w^* > 0$ implies $(x^*,w^*) \in C^{n+1}$.

We now show that x^* is a stationary point of (f,D^n). We have two cases:

Case I. $(x^*,w^*) \in C^{n+1}$. In this case, let $z = 0$, and let $y = f(x^*)$. $(x^*,w^*) \in C^{n+1}$ implies $f(x^*) \geq 0$, and so $y \geq 0$. Also, $x_i^* > 0$ implies $f_i(x^*) \leq 0$, but $f(x^*) \geq 0$, therefore $f_i(x^*) = 0$. Thus $x\cdot y = x\cdot f(x) = 0$. Finally, $z(1 - e^T\cdot x^*) = 0$ since $z = 0$. Therefore, x^* is a stationary point.

Case 2. $(x^*,w^*) \notin C^{n+1}$. Let $z = -\min\{f_1(x^*), \dots, f_n(x^*)\}$. Note $z \geq 0$. Let $y = f(x^*) + ze$. We have $-z \leq f_i(x^*)$ for each i. Hence $y = f(x^*) + ze \geq 0$. Furthermore, $x_i^* > 0$ implies $f_i(x^*) \leq f_j(x^*)$ for any j, and since $f(x^*) \not\geq 0$, $f_i(x^*) = -z$. Therefore $y_i = 0$, and so $x^*\cdot y = 0$. Finally, since $(x^*,w) \notin C^{n+1}$, we must have $w^* = 0$, so

$e^T x^* = 1$, which implies $z(1 - e^T x^*) = 0$. Thus x^* is a stationary point of (f, D^n). $\otimes$

Our proof suggests an algorithm for computing x^*. Choose a triangulation of S^{n+1} with small diameter and compute a simplex x which is completely labelled in its face by following the path starting at $\emptyset \in B$ (see Section 4.2). Any point in x almost satisfies the conclusion of Theorem 5.9, and hence is an approximate stationary point. In Section 5.3, we suggest an algorithm for finding points guaranteed by Theorem 5.9, using a "homotopy" principle.

We conclude this section with two theorems that follow directly from the covering theorem, Theorem 5.10. The set-up for these theorems is as follows:

Let A be an $n \times (n+1)$ matrix such that the $A\lambda = y$, $\lambda \geq 0$, has a solution for any $y \in \mathbb{R}^n$ (i.e. the cone space of the columns of A is $\mathbb{R}^n$). We state without proof the following:

<u>Remark</u>: For fixed y, c, $y = c + A\lambda$, $\lambda \geq 0$ has a unique semi-positive solution where by semi-positive we mean nonnegative and not positive. $\otimes$

Let S be an n-simplex in $\mathbb{R}^n$ and let $f(\cdot): S \to \mathbb{R}^n$ be continuous. For each $y \in \mathbb{R}^n$, let $\lambda(y)$ be the unique non-positive solution λ to $y = c + A\lambda$, $\lambda \geq 0$. We have the following theorem:

Theorem 5.12. For $x \in S$, let $\alpha(x)$ be the barycentric representation of x. Then there exists a point $x^* \in S$ such that $\alpha(x^*)\cdot\lambda(f(x^*)) = 0$.

PROOF: Let $C^i = \{x \in S | \lambda_i(f(x)) = 0\}$ for $i = 0,\ldots,n$. Each $x \in S$ is an element of at least one C^i. Furthermore, since f is continuous and $\lambda(\cdot)$ is continuous, C^i is closed for each i. Applying Theorem 5.10, we have that there is an $x^* \in S$ such that $\alpha_i(x^*) > 0$ implies $x^* \in C^i$. Therefore, $\alpha(x^*)\cdot\lambda(f(x^*)) = 0$. $\otimes$

Theorem 5.13. For $x \in S$, let $\alpha(x)$ be the barycentric representation of x. Then there exists at least one point $x^* \in S$ such that

$$\{i|\alpha_i(x^*) > 0\} \subset \{i|f(x^*)^T A_{\cdot i} \leq c^T A_{\cdot i}\} \quad .$$

PROOF: For $i = 0,\ldots,n$, let $C^i = \{x \in S|[f(x)]^T A_{\cdot i} \leq c^T A_{\cdot i}\}$. I claim that $\cup_{i=0}^n C^i \supset S$. Suppose not. Then there is an $x \in S$ such that $f(x)^T A > c^T A$. Let $\pi = c - f(x)$. We have $\pi A < 0$. We have $\pi = A\lambda$ for some $\lambda \geq 0$. Therefore $0 \geq \pi A \lambda = \pi\cdot\pi \geq 0$. Therefore $\pi = 0$. But then $f(x)^T A = c^T A$, a contradiction. Thus $\cup_{i=0}^n C^i \supset S$. Applying Theorem 5.10, we have that there exists $x^* \in S$ such that

$$\{i|\alpha_i(x^*) > 0\} \subset \{i|f(x^*)^T A_{\cdot i} \leq c^T A_{\cdot i}\} \ . \qquad \otimes$$

5.3. A Homotopy Algorithm for Computing x^* of Theorem 5.9

In Section 5.2, we showed that the covering theorem, Theorem 5.9, can be used to prove three other theorems, including the existence of stationary points. The covering theorem is proved by applying the Generalized Sperner Lemma in a limiting sense. To compute the point x^* of the covering theorem, one could set up a triangulation of small diameter, and follow the path from $\emptyset \in B$, as suggested in Section 4.2, to its other endpoint. Any vector in the endpoint simplex approximates the properties of x^*. The problem with this technique is that the number of simplices en[illegible] in the path may be very large. Eaves [4] an[illegible] Saigal [illegible] introduced the idea of a homotopy meth[illegible]-following methods, and computational experience has proven the superior nature of homotopy methods in path-following algorithms (see Wilmuth [43]).

In this section, we present a homotopy approach to calculating a point x^* of Theorem 5.9. Let $S^n = \{x \in \mathbb{R}^n \mid e^T x = 1, x \geq 0\}$, the unit simplex in $\mathbb{R}^n$. Let C be a locally finite triangulation of $S^n \times [0,\infty)$ with the property that for every $\epsilon > 0$, there is a $t > 0$ such that any simplex of C in $S^n \times [t,\infty)$ has diameter less than ϵ. Such triangulations exist and specific pivot rules have been calculated for some of these (see Eaves [4], Todd [41], and van der Laan and Talman [26]). Let K^0 denote the vertices of C.

Let C^i, $i = 1,\ldots,n$, be closed sets in $\mathbb{R}^n$ such that $\bigcup_{i=1}^n C^i \supset S^n$. Each vertex of K^0 can be written as (v,t) where $v \in S^n$ and $t \in [0,\infty)$. For each $(v,t) \in K^0$, let $L(v,t) =$ some i such that $v \in C^i$.

Our next step is to set up a V-complex on $S^n \times [0,\infty)$. Let $N = \{1,\ldots,n\}$ and let $\mathfrak{S} = \{S \mid S \subset N\}$. For $T = \emptyset$, let $A(T) = \{\emptyset, \{(e^j,0)\}\}$ for some fixed $j \in N$; for $T \neq \emptyset$, $T \in \mathfrak{S}$, let $A(T)$ be the pseudomanifold corresponding to the restriction of C to

$$\{(x,0) \in S^n \times [0] \mid i \notin T \cup \{j\} \text{ implies } x_i = 0\}$$

$$\cup \{(x,t) \in S^n \times (0,\infty) \mid i \notin T \text{ implies } x_i = 0\}$$

Figures 5.4 and 5.5 illustrate this construction for $n = 2$ and 3, respectively, with $j = 1$. In Figure 5.5, the triangulation has been omitted to make the figure more understandable.

For $j \in T$, $A(T)$ corresponds to a nonempty closed convex set. For $T \neq \emptyset$, $j \notin T$, $A(T)$ corresponds to the union of two nonempty closed convex sets that share a common boundary.

It is simple to show that $A(\cdot)$, $\mathfrak{S}$, and K satisfy the conditions of a V-complex, where K is the complex (an n-psuedomanifold itself) corresponding to C.

Let us now examine the sets G and B. G is empty, since $\mathfrak{S} = \{T \mid T \subset N\}$. As far as B is concerned, $\emptyset \in B$, since $A(\emptyset)$ contains only one 0-simplex, $\{(e^j,0)\}$. For $T \neq \emptyset$, $\partial' A(T)$ is empty, and so the only element of B is $\emptyset$. Thus, the path starting from $\emptyset \in B$ has no other endpoint, and so must contain an infinite number of simplices. Let $\emptyset = x_0, x_1, x_2, \ldots,$ be the path of adjacent simplices whose endpoint x_0 is $\emptyset$. Then for each $\epsilon > 0$, there is an $m > 0$ such that for all $i > m$ the diameter of x_i is less than ϵ. Thus the simplices of the path get smaller and smaller in the limit. Let x_i have diameter

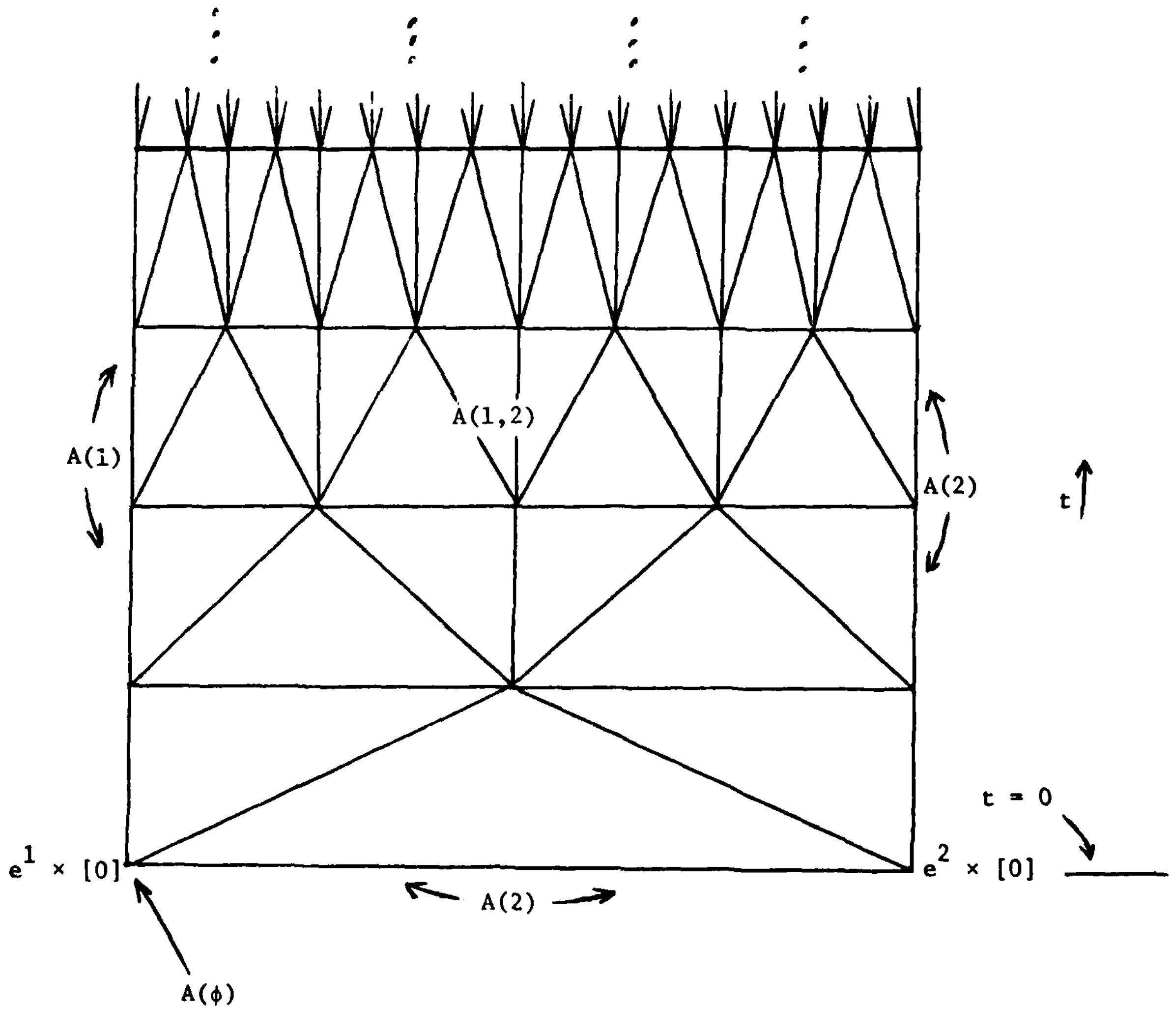
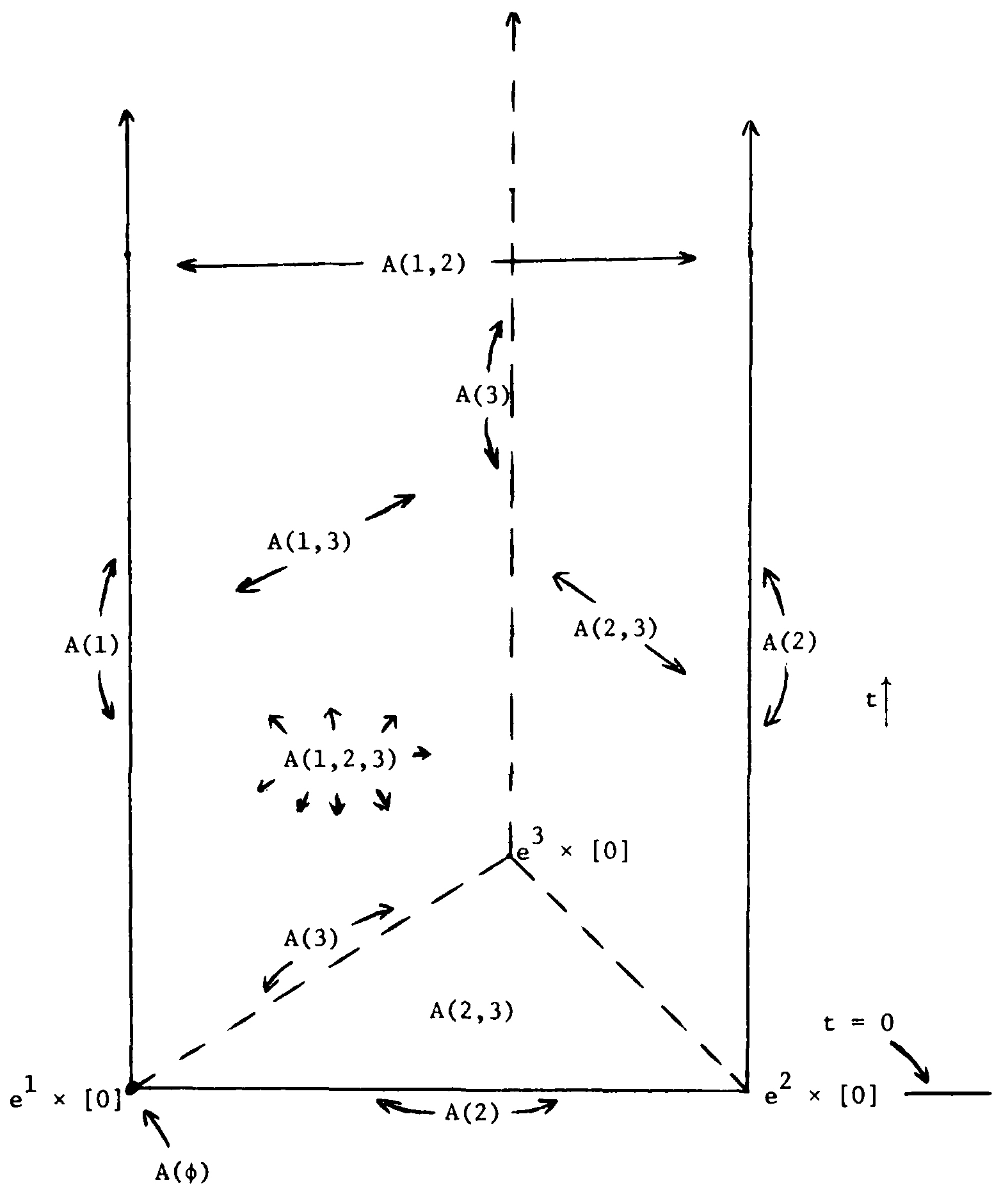


Figure 5.4



A(1,2)
A(3)
A(1,3)
A(1)
A(2,3)
A(2)
t
A(1,2,3)
$e^3 \times [0]$
A(3)
A(2,3)
t = 0
$e^1 \times [0]$
A(2)
$e^2 \times [0]$
A(ϕ)

Figure 5.5

less than ϵ. Then choose a vector (v,t) in x_i. v then approximately satisfies the conditions of Theorem 5.9.

The homotopy algorithm consists of following the path $\emptyset = x_0, x_1, x_2, \ldots$.

5.4. Extensions of Tucker's Lemma

In this section we make some brief remarks concerning the relationship between Tucker's Lemma and antipodal point theorems. Two established antipodal point theorems are the Borsuk-Ulam Theorem and the Lusternik-Schnirelman Theorem (see Lefschetz [27]).

Borsuk-Ulam Theorem. Let $B^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$, and let $f(\cdot): B^{n-1} \to \mathbb{R}^{n-1}$ be continuous. Then there exists $x^* \in B^{n-1}$ such that $f(x^*) = f(-x^*)$.

Lusternik-Schnirelman Theorem: Let $B^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$, and let C^i, $i = 1,\ldots,n$, be closed sets in $\mathbb{R}^n$ such that $\bigcup_{i=1}^{n} C^i \supset B^{n-1}$. Then there is an $i \in \{1,\ldots,n\}$ and an $x^* \in B^{n-1}$ such that both x^* and $-x^*$ are elements of C^i.

We have the following relationships:

Lemma 5.14. Tucker's Lemma implies the Borsuk-Ulam Theorem.

Lemma 5.15. Tucker's Lemma implies the Lusternik-Schnirelman Theorem.

The proofs of these two lemmas can be found in Tucker [42], or in terminology more familiar to operations researchers, in Freund and Todd [11].

We also have:

Lemma 5.16. The Borsuk-Ulam Theorem implies Tucker's lemma.

PROOF: Let $C^n = \{x \in \mathbb{R}^n \mid -\rho e \leq x \leq \rho e\}$ and let C be the Union-Jack triangulation J^1 of $\mathbb{R}^n$ (see Todd [41]) restricted to C^n. The vertex set K^0 of C is the set of integral points of C. Let $L(\cdot):K^0 \to \{1,\ldots,n,-1,\ldots,-n\}$ be a labelling function which is odd on the boundary of C. For each $v \in K^0$, define $f(v) = \text{sign}(L(v))\cdot e^{(L(v))}$ and extend $f(\cdot)$ in a piece-wise linear manner over all of C^n. Note that $f(\cdot)$ is continuous, and since J^1 is symmetric, $f(\cdot)$ is odd on the boundary of C^n.

Let $B^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}$, let $B^{n+} = \{x \in B^n \mid x_{n+1} \geq 0\}$, and let $B^{n-} = \{x \in B^n \mid x_{n+1} \leq 0\}$. Let $g:B^{n+} \to C^n$ be the following map:

$$g(x) = \begin{cases} \dfrac{\rho(x_1,\ldots,x_n)\ \|x_1,\ldots,\ x_n\|_2}{\|(x_1,\ldots,x_n)\|_\infty}, & (x_1,\ldots,x_n) \neq 0. \\ \\ 0, & (x_1,\ldots,x_n) = 0\ . \end{cases}$$

Note that $g(\cdot)$ is bicontinuous and onto. For $x \in B^n$, let

$$h(x) = \begin{cases} f\circ g(x)\ , & x \in B^{n+} \\ \\ -f\circ g(x)\ , & x \in B^{n-} \end{cases}$$

$h(\cdot)$ is an odd continuous function from B^n into $\mathbb{R}^n$. By the Borsuk-Ulam Theorem, there exists x^* such that $h(x^*) = h(-x^*)$. Without loss of generality, we may assume $x^* \in B^{n+}$. Thus $h(x^*) = 0$, whereby $f \circ g(x^*) = 0$. Setting $\bar{x} = g(x^*)$, we see there exists $\bar{x} \in C^n$ such that $f(\bar{x}) = 0$. $f(\bar{x}) = \Sigma\, \lambda_i\, \text{sign}(L(v^i)) \cdot e^{|L(v^i)|}$ for appropriate v^i and $\lambda_i \geq 0$. Thus there must be a pair of vertices v^{i_1}, v^{i_2} such that $L(v^{i_1}) = -L(v^{i_2})$, proving Tucker's Lemma. ⊗

CHAPTER VI

ORIENTATION AND H-COMPLEXES

6.0. Introduction

In this chapter, we return to the abstract setting of V- and H-complexes, and deal with orientation of H-complexes on a deeper level. Assuming an H-complex is orientable, we show how to pivot on adjacent simplices in a way that preserves certain properties of the orientation of the simplices. We are thus able to give further insight into the paths and endpoints of paths on H-complexes. In the last section we give mild sufficient conditions that ensure the orientability of an H-complex.

The use of orientation in complementary pivot schemes was first developed by Shapley [38] for the linear complementarity problem, advanced by Eaves and Scarf [8] and Eaves [6] for subdivided manifolds, and extended to pseudomanifolds by Lemke and Grotzinger [30]. Our set-up is slightly different than that of Lemke and Grotzinger; however, the interested reader can easily establish the similarity.

6.1. Pivots and C-Pivots on Pseudomanifolds

Let $\bar{K}$ be an orientable H-complex of dimension n, oriented by $Or(\cdot)$, with vertex set $\bar{K}^0$. Let $N = \{1,\ldots,n\}$ and let $L(\cdot):K^0 \to N$. We define the set

$$D = \{\bar{x} \in \bar{K} \,\big|\, |\bar{x}| = n+1,\ L(\bar{x}) = N\} \cup \{\bar{x} \in \partial\bar{K} \,\big|\, |\bar{x}| = n,\ L(\bar{x}) = N\}.$$

D consists then of n-simplices of $\bar{K}$ whose labels exhaust N, and simplicies on the boundary of $\bar{K}$ whose labels exhaust N. We remark that the two sets above whose union is D are disjoint. Let these sets be D_1 and D_2, respectively.

Let $\bar{x} \in D$. There is a very natural way to order the elements of $\bar{x}$. If $\bar{x} \in D_1$, we can write $\bar{x} = \{v_0, \ldots, v_n\}$. The ordering $(v_{i_0}, \ldots, v_{i_n})$ of $\bar{x}$ is called a C-ordering if and only if:

$$L(v_{i_j}) = j, \qquad j = 1, \ldots, n .$$

Note there are always two orderings of $\bar{x}$. The reason for this is that among the labels of $\bar{x}$, there is some unique $r \in N$ such that two vertices of $\bar{x}$ have the label r. For $j \in N \setminus \{r\}$, the $j^{\underline{th}}$ component of a C-ordering of $\bar{x}$ must be the unique vertex $v_{i_j} \in \bar{x}$ for which $L(v_{i_j}) = j$. Denote by v' and v'' those two vertices in $\bar{x}$ whose labels are r. Then the two C-orderings of $\bar{x}$ are:

$$(v'', v_{i_1}, \ldots, v_{i_{r-1}}, v', v_{i_{r+1}}, \ldots, v_{i_n})$$

and

$$(v', v_{i_1}, \ldots, v_{i_{r-1}}, v'', v_{i_{r+1}}, \ldots, v_{i_n})$$

Also note that these two orderings have opposite orientations, i.e. one is (+) and the other is (-).

If $\bar{x} \in D_2$, we can write $\bar{x} = \{v_1, \ldots, v_n\}$. The ordering $(v_{i_1}, \ldots, v_{i_n})$ is called a C-ordering if and only if

$$L(v_{i_j}) = j, \qquad j = 1, \ldots, n.$$

The C-ordering for $\bar{x} \in D_2$ is unique.

With the notion of a pivot in mind, we now define a C-pivot on elements of D. For $x \in D_1$, let $(v_{i_0}, \ldots, v_{i_n})$ be a C-ordering of $\bar{x}$. A C-pivot is performed on $\bar{x}$ as follows:

Case 1. $\{v_{i_1}, \ldots, v_{i_n}\} \in \partial\bar{K}$. In this case, simply drop v_{i_0} from $\bar{x}$, and let $\bar{y} = \{v_{i_1}, \ldots, v_{i_n}\}$. The derived ordering on $\bar{y}$ is $(v_{i_1}, \ldots, v_{i_n})$.

Case 2. $\{v_{i_1}, \ldots, v_{i_n}\} \notin \partial\bar{K}$. In this case, there is a unique $\bar{v} \in \bar{K}^0$ such that $\{v_{i_1}, \ldots, v_{i_n}, \bar{v}\} \in \bar{K}$. $L(\bar{v}) = r$ for some $r \in N$. Set $\bar{y} = \{v_{i_1}, \ldots, v_{i_n}, \bar{v}\}$ and form the new ordering $(v_{i_r}, v_{i_1}, \ldots, v_{i_{r-1}}, \bar{v}, v_{i_{r+1}}, \ldots, v_{i_n})$ of $\bar{y}$.

If $\bar{x} \in D_2$, let $(v_{i_1}, \ldots, v_{i_n})$ be the C-ordering of $\bar{x}$. A C-pivot on $\bar{x}$ is performed as follows:

Let $\bar{v}$ be the unique element of $\bar{K}^0$ such that $\bar{x} \cup \{\bar{v}\}$ is an n-simplex of $\bar{K}$. $L(\bar{v}) = r$ for some $r \in N$. Set $\bar{y} = \{v_{i_1}, \ldots, v_{i_n}, \bar{v}\}$ and from the new ordering $(v_{i_r}, v_{i_1}, \ldots, v_{i_{r-1}}, \bar{v}, v_{i_{r+1}}, \ldots, v_{i_n})$ of $\bar{y}$.

We have the following results on C-pivots:

Proposition 6.1. Let $\bar{y}$ be derived from a C-pivot on $\bar{x} \in D_1$. Then the ordering of $\bar{y}$ is a C-ordering and the orderings on $\bar{x}$ and $\bar{y}$ as specified in the C-pivot have the same orientation.

PROOF: The first conclusion of the proposition follows immediately from the ordering defined on $\bar{y}$. The second conclusion follows from a case analysis.

Case 1. $\bar{y} \in \partial\bar{K}$. Then $Or(v_{i_0}, \ldots, v_{i_n}) = (-1)^0 \, Or(v_{i_1}, \ldots, v_{i_n}) = Or(v_{i_1}, \ldots, v_{i_n})$.

Case 2. $\bar{y} \notin \partial\bar{K}$. Then

$$Or(v_{i_0}, \ldots, v_{i_n}) = -Or(\bar{v}, v_{i_1}, \ldots, v_{i_n})$$

$$= Or(v_{i_r}, v_{i_1}, \ldots, v_{i_{r-1}}, \bar{v}, v_{i_{r+1}}, \ldots, v_{i_n}) . \quad \otimes$$

Proposition 6.2. Let $\bar{y}$ be derived from a C-pivot on $\bar{x} \in D_2$. Then the ordering of $\bar{y}$ is a C-ordering and the orderings on $\bar{x}$ and $\bar{y}$, as specified in the C-pivot, have opposite orientation.

PROOF: The first conclusion follows directly from the ordering fixed on y. For the second conclusion, note that

$$Or(v_{i_r}, v_{i_1}, \ldots, v_{i_{r-1}}, \bar{v}, v_{i_{r+1}}, \ldots, v_{i_n})$$

$$= -Or(\bar{v}, v_{i_1}, \ldots, v_{i_n}) = -Or(v_{i_1}, \ldots, v_{i_n}) . \quad \otimes$$

6.2. Orientations on Paths Generated by C-Pivots

In Section 3.5, we characterized paths generated by a labelling $L(\cdot)$ on H-complexes. In this section, we show the connection between C-pivots and these paths, and prove a result on orientation along paths.

Let $\bar{K}$ be an orientable H-complex oriented by $Or(\cdot)$, $\bar{K}^0$ its vertex set, and assume, without loss of generality, that $N = \{1,\dots,n\}$. Let $L(\cdot):\bar{K}^0 \to N$ be a labelling function. Let $\langle \bar{x} \rangle_i$ be a path on $\bar{K}$, possibly without left and/or right endpoints.

Choose $\bar{x}$ an element of the path. Note that $L(\bar{x}) = N$. If $\bar{x}$ is an endpoint of the path (say a left endpoint, and we can assume $\bar{x} = \bar{x}_0$, without loss of generality), there is a unique C-ordering of $\bar{x}_0$. Let $\bar{y}$ be derived from $\bar{x}_0$ by a C-pivot on $\bar{x}_0$. Then $\bar{y} = \bar{x}_1$, and $Or(\bar{x}_1) = -Or(\bar{x}_0)$ from Proposition 6.2. We can keep performing C-pivots $\bar{x}_i$, until we reach the right endpoint of the path, if it exists. For each of these pivots, we have $Or(\bar{x}_{i+1}) = Or(\bar{x}_i)$ by Proposition 6.1. We have just proved the following

Lemma 6.3. Let $\langle \bar{x}_i \rangle_i$ be a path with left endpoint $\bar{x}_0$. If $\bar{x}_{i+1}$ is obtained from $\bar{x}_i$ by a C-pivot, $Or(\bar{x}_0) = -Or(\bar{x}_i)$ for all $i > 0$. ⊗

In particular, we have

Corollary 6.4. Let $\langle \bar{x}_i \rangle_i$ be a path with left- and right-endpoints $\bar{x}_0$ and $\bar{x}_m$, generated by a series of C-pivots starting at $\bar{x}_0$. Then

i) $\mathrm{Or}(\bar{x}_0) = -\mathrm{Or}(\bar{x}_m)$,

and

ii) $\mathrm{Or}(\bar{x}_i) = \mathrm{Or}(\bar{x}_j)$ for all $0 < i, j \leq m$.

Corollary 6.4 is analogous to other path orientation theorems presented elsewhere, see, for example, Shapley [38], Eaves and Scarf [8], Eaves [6], and Lemke and Grotzinger [30]. All of these theorems assert that the orientation along a path is constant except at the endpoints, whose orientations are opposite in sign.

Now suppose that $\bar{x}_i$ is an element of the path $\langle \bar{x}_i \rangle_i$ and $\bar{x}_i$ is not an endpoint. Then $\bar{x}_i \in D_1$. Since $L(\bar{x}) = N$, we can choose two C-orderings of $\bar{x}$, each one opposite in sign. C-pivoting on one of these orderings will yield $\bar{x}_{i+1}$, and the C-ordering of $\bar{x}_{i+1}$ will have the same orientation as the C-ordering of $\bar{x}_i$. Continuing the C-pivot process, we will generate the path elements $\bar{x}_i, \bar{x}_{i+1}, \bar{x}_{i+2}, \ldots$, terminating if and only if $\langle \bar{x} \rangle_i$ has a right endpoint. By Proposition 6.1, $\mathrm{Or}(\bar{x}_j) = \mathrm{Or}(\bar{x}_i)$ for all $j > i$. A parallel argument for the other C-ordering completes the proof of the following

<u>Lemma 6.5</u>. Let $\langle \bar{x}_i \rangle_i$ be a path on $\bar{K}$ and let $\bar{x}_i$ be an element of this path that is not an endpoint. Let the entire path be generated from $\bar{x}_i$ by its two C-orderings. We have $\mathrm{Or}(\bar{x}_j) = -\mathrm{Or}(\bar{x}_k)$ for all $j < i < k$. ⊗

In particular, we have

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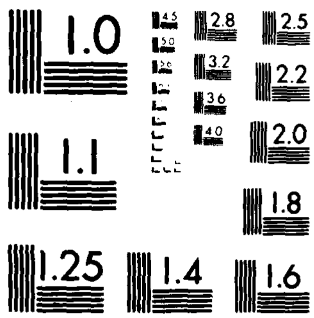

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<u>Corollary 6.6</u>. Let $\langle x_i \rangle_i$ be a path on $\bar{K}$ with left and right endpoints $\bar{x}_0$ and $\bar{x}_m$, respectively. If this path is generated from $\bar{x}_i$, $0 < i < m$, by the two C-orderings of $\bar{x}_i$, then

i) $Or(\bar{x}_0) = -Or(\bar{x}_m)$,

and

ii) $Or(\bar{x}_j) = -Or(\bar{x}_k)$ for all $j < i < k$.

Corollaries 6.4 and 6.6 are illustrated schematically in Figures 6.1 and 6.2.

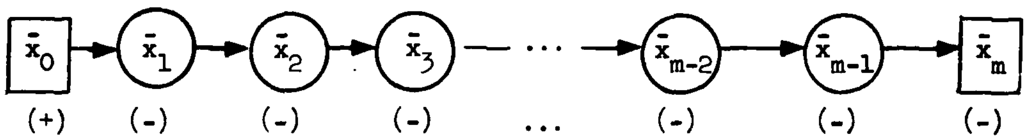

Figure 6.1

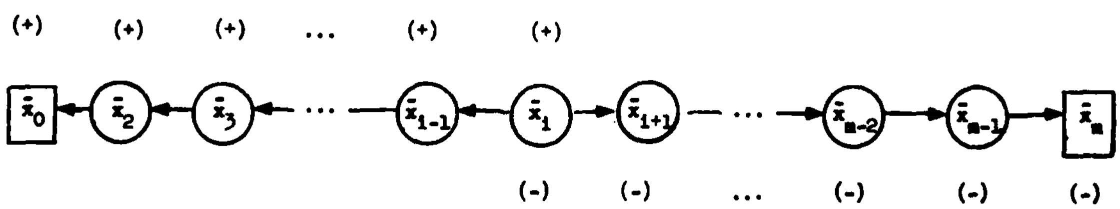

Figure 6.2

By way of concluding Sections 6.1 and 6.2, we remark that the usual path orientation results for manifolds carry over to H-complexes. Actually, they do more than this--they carry over to orientable n-pseudomanifolds. For the only properties of H-complexes used in these two sections was that $\bar{K}$ is an orientable n-pseudomanifold and that the label set N contains n elements.

6.3. Conditions for which an H-complex is Orientable

In this section we give conditions on $\bar{K}$ that guarantee that $\bar{K}$ is orientable. Let K, $\mathfrak{I}$, and $A(\cdot)$ define a V-complex, and let $\bar{K}$ be the H-complex associated with K. Let $|N| = n$. Assume that

i) for each $T \in \mathfrak{I}$, $A(T)$ is locally finite and orientable, and hence homogeneous, and

ii) for all $S, T \in \mathfrak{I}$, $S \subset T$, there is a sequence $i_1,\ldots,i_m$, such that $S \cup \{i_1,\ldots,i_k\} \in \mathfrak{I}$, $k = 1,\ldots,m$, and $S \cup \{i_1,\ldots,i_m\} = T$.

We will show that if K, $\mathfrak{I}$, and $A(\cdot)$ satisfy the above two assumptions, then $\bar{K}$ is orientable.

In Section 6.4, we will discuss ways to test assumptions (i) and (ii) for specific V-complexes. Obviously, verifying condition (ii) is very straightforward. Condition (i), however, requires special attention. Our first result is:

Proposition 6.7. For all $S, T \in \mathfrak{S}$, there is a sequence $S_0, \dots, S_m$ such that $S_i \in \mathfrak{S}$, $i = 0,\dots,m$, $S_0 = S$, $S_m = T$, and $|S_{i-1} \triangle S_i| = 1$, $i = 1,\dots,m$.

PROOF: If $S = T$, the conclusion is trivial. If $S \neq T$, $S \cap T \in \mathfrak{S}$. We can write

$$S = S \cap T \cup \{s_1,\dots,s_k\}, \quad T = S \cap T \cup \{t_1,\dots,t_\ell\}$$

for appropriate k, ℓ, s_i, and t_j. We can assume as well that the s_i are distinct and that the t_j are distinct. By assumption (ii), we can assume that $S \cap T \cup \{s_1,\dots,s_i\} \in \mathfrak{S}$ for $i \leq k$, and $S \cap T \cup \{t_1,\dots,t_j\} \in \mathfrak{S}$ for $j \leq \ell$. Then the sequence

$$S,\ S \setminus \{s_k\},\ S \setminus \{s_k, s_{k-1}\}, \dots,\ S \cap T,\ S \cap T \cup \{t_1\},\ S \cap T \cup \{t_1,t_2\}, \dots,\ T$$

is a sequence of elements of $\mathfrak{S}$, and successive members S_{i-1}, S_i satisfy $|S_{i-1} \triangle S_i| = 1$. $\otimes$

Towards proving our main result, we make the following:

Definition. For $T \in \mathfrak{S}$, define

$$\bar{A}(T) = \{x \cup Q \mid x \in A(T),\ Q \subset Q_T,\ x \cup Q \neq \emptyset\} .$$

$\bar{A}(T)$ can be thought of as a conical construction of $A(T)$ with each q_i, $i \notin T$. We have:

<u>Lemma 6.8</u>. $\bar{A}(T)$ is an orientable n-pseudomanifold.

<u>PROOF</u>: Clearly $\bar{A}(T)$ is closed under nonempty subsets. Let $x \cup Q \in \bar{A}(T)$. Then there is a y in $A(T)$ such that $T_y = T$, since $A(T)$ is a pseudomanifold. Then $x \cup Q \subset y \cup Q_T$, and $|y \cup Q_T| = |T| + 1 + n - |T| = n+1$.

Any n-simplex of $\bar{A}(T)$ is of the form $y \cup Q_T$, where y is a $|T|$-simplex in $A(T)$. Let $x \cup Q$ be an (n-1)-simplex in $\bar{A}(T)$, that is a subset of $y \cup Q_T$. Suppose $x \cup Q \subset z \cup Q_z$, $|z \cup Q_z| = n+1$, and $z \cup Q_z \neq y \cup Q_T$. But then $Q_z = Q_T$, and since $A(T)$ is a pseudomanifold, the choice of z is unique. This proves $\bar{A}(T)$ is an n-pseudomanifold.

Now let $x \cup Q_T$ and $y \cup Q_T$ be n-simplices in $\bar{A}(T)$. Then x and y are $|T|$-simplices in $A(T)$. Since $A(T)$ is homogeneous, there is a sequence $x = s_1, s_2, \ldots, s_m = y$ of $|T|$-simplices in $A(T)$ such that $|s_i \cap s_{i+1}| = |T|$, $i = 1,\ldots,m-1$. Then $x \cup Q_T = s_1 \cup Q_T$, $s_2 \cup Q_T$, $\ldots$, $s_m \cup Q_T = y \cup Q_T$ is a sequence of n-simplices in $\bar{A}(T)$ and $|(s_i \cup Q_T) \cap (s_{i+1} \cup Q_T)| = n$, $i = 1,\ldots,m-1$. Therefore $\bar{A}(T)$ is homogeneous.

Finally, we show that $\bar{A}(T)$ is orientable. Let $Or(\cdot)$ be a coherent orientation of $|T|$-simplices of $A(T)$. Let $x \cup Q_T$ be an n-simplex of $\bar{A}(T)$. Let $|T| = t$. Order the vertices of $x \cup Q_T$, as $(v_0,\ldots, v_n)$, and let p be the number of transpositions needed to "push" those $v_i \in Q_T$ to the last n-t places of the ordering, while preserving the local ordering of those $v_i \in A(T)$ and the local ordering of those $v_i \in Q_T$. Then we define $Or(v_0,\ldots,v_n) = (-1)^p \, Or(x)$.

If $y \cup Q_T$ is obtained from x by a pivot, $Or(y \cup Q_T) = (-1)^p\, Or(y)$ $= (-1)^p\,(-Or(x)) = -Or(x \cup Q_T)$. Thus $\bar{A}(T)$ is orientable. $\otimes$

<u>Lemma 6.9</u>. $\bar{K} = \bigcup_{T \in \mathfrak{T}} \bar{A}(T)$.

<u>PROOF:</u> Let $\bar{x} \in K$. Then we can write $\bar{x} = x \cup Q$ where $x \in K$, and $Q \subset \bar{Q}$, and $Q \subset Q_{T_x}$. But then $\bar{x} \in \bar{A}(T_x)$. Conversely, let $x \cup Q \in \bar{A}(T)$. Then $Q \subset Q_{T_x}$ and so $x \cup Q \in \bar{K}$. $\otimes$

<u>Lemma 6.10</u>. Any n-simplex of $\bar{K}$ is an element of exactly one $\bar{A}(T)$.

<u>PROOF:</u> Let $x \cup Q$ be an n-simplex in $\bar{K}$. Then $Q = Q_{T_x}$ and x is full. Thus $x \cup Q \in \bar{A}(T_x)$. Suppose $x \cup Q \in \bar{A}(S)$ for some $S \in \mathfrak{T}$. Then $x \in A(S)$ and hence $S \supset T_x$. Also $Q_{T_x} \subset Q_S$ which implies $S \subset T_x$. Thus $S = T_x$. $\otimes$

Thus we see that as T ranges over all elements of $\mathfrak{T}$, the $\bar{A}(T)$ partition $\bar{K}$ into "disjoint" n-pseudomanifolds. We use disjoint cautiously since this partitioning only takes place among the n-simplices of $\bar{K}$.

Next we have

<u>Proposition 6.11</u>. $\bar{K}$ is homogeneous.

<u>PROOF:</u> Let $\bar{x}$ and $\bar{y}$ be n-simplices in $\bar{K}$. We can write $\bar{x} = x \cup Q_{T_x}$, $\bar{y} = y \cup Q_{T_y}$ for appropriate $x, y \in K$. By Proposition 6.7, there is a

sequence $T_x = T_1, \ldots, T_m = T_y$ such that $T_i \in \mathfrak{J}$, $i = 1,\ldots,m$, and $|T_i \triangle T_{i+1}| = 1$, $i = 1,\ldots,m-1$.

We shall now show how to construct a sequence of neighboring simplices in $\bar{K}$ that have $\bar{x}$ and $\bar{y}$ as endpoints, using an induction argument on m. If $m = 1$, then such a sequence of neighboring simplices exists because $\bar{x}, \bar{y} \in \bar{A}(T) \subset \bar{K}$, and $\bar{A}(T)$ is homogeneous. Suppose a sequence of neighboring simplices exists $\langle s_i \rangle_{i=0}^{\ell}$ whose endpoints are $\bar{x}$ and $\bar{z} \in T_{m-1}$. Then either $T_m = T_{m-1} \cup \{k\}$ for some $k \notin T_{m-1}$, or $T_m = T_{m-1} \setminus \{k\}$ for some $k \in T_{m-1}$. In the former case, $\tilde{z} = \bar{z} \setminus \{q_k\} \cup \{w\}$ is an n-simplex in $\bar{K}$, that is in $\bar{A}(T_m)$, for some unique $w \in \bar{K}^0$. Since $\bar{A}(T_m)$ is homogeneous, there is a sequence of neighboring n-simplices $\langle t_i \rangle_{i=0}^{j}$, where $t_0 = \tilde{z}$, $t_j = \bar{y}$. Thus the sequence

$$\bar{x} = s_0, \ldots, s_\ell = \bar{z}, \tilde{z} = t_0, \ldots, t_j = \bar{y}$$

of neighboring simplices has $\bar{x}$ and $\bar{y}$ as its endpoints. An analogous argument establishes the result when $T_m = T_{m-1} \setminus \{k\}$. ⊗

The next few results will also be used in the proof that $\bar{K}$ is orientable.

<u>Proposition 6.12</u>. There is a unique set $T \in \mathfrak{J}$ such that $S \in \mathfrak{J}$ implies $S \supset T$.

<u>PROOF</u>: Define $T^* = \bigcap_{S \in \mathfrak{J}} S$. Then $T^* \in \mathfrak{J}$ and any $S \in \mathfrak{J}$ contains T^*. Clearly T^* is uniquely determined. ⊗

Proposition 6.13. Let $S, T \in \mathfrak{I}$, $S \neq T$, $|S| = |T|$. Then $A(S) \cap A(T)$ contains no (n-1)-simplices.

PROOF: Let $x \cup Q \in \bar{A}(S) \cap \bar{A}(T)$. Let $t = |S| = |T|$. We have $x \in A(S)$, $x \in A(T)$, so $x \in A(S \cap T)$. But $|S \cap T| \leq t-1$. Thus $|x| \leq 1$. Also $Q \subset Q_S \cap Q_T = Q_{S \cup T} \leq n - (t+1)$. Thus $|x \cup Q| \leq t - 1 + n - t - 1$ $\leq n - 2$. Therefore $x \cup Q$ cannot be an (n-1)-simplex. ⊗

We are now ready to describe an inductive procedure for orienting $\bar{K}$. Let $T^* \in \mathfrak{I}$ be the set described in Proposition 6.12. Let $d = |T^*|$. Let $m = \max_{T \in \mathfrak{I}} |T| - d$. Then we partition $\mathfrak{I}$ into m+1 classes, $\mathfrak{I}_d, \ldots, \mathfrak{I}_{d+m}$, where $\mathfrak{I}_k = \{T \in \mathfrak{I} \,\big|\, |T| = d+k\}$. Note that $\mathfrak{I} = \cup_{k=0}^{m} \mathfrak{I}_{d+m}$, and *for all* $k \neq j$, $\mathfrak{I}_k \cap \mathfrak{I}_j = \emptyset$. *Our procedure for orienting* $\bar{K}$ *is* as follows.

Step 0. Orient $\bar{A}(T^*)$. Let $Or(\cdot)$ denote the orientation on $\bar{A}(T^*)$.

Set $\bar{K}_0 = \bar{A}(T^*)$.

Step i (i = 1,...,m): Let $\bar{K}_i = \bar{K}_{i-1} \cup (\cup_{T \in \mathfrak{I}_i} \bar{A}(T))$. Extend the orientation $Or(\cdot)$ to $\bar{K}_i$ by using the induced orientation on $\partial\bar{K}_{i-1}$ to orient $\partial\bar{A}(T)$, $T \in \mathfrak{I}_i$. We now show that each step of this procedure is executable and the result is a coherent orientation of $\bar{K}$. Note $\bar{K}_m = \bar{K}$. Our proof is as follows:

Clearly Step 0 is executable, since $\bar{A}(T^*)$ is orientable. Suppose steps $0,\ldots, i-1$ are executable and result in a coherent orientation of $\bar{K}_{i-1}$. The following lemma serves as a basis for our proof:

Lemma 6.14. Suppose $T' \subset \mathfrak{S}_i$. Then $\bar{K}_{i-1} \cap \bar{A}(T')$ is an orientable $(n-1)$-pseudomanifold and is a subset of $\partial\bar{K}_{i-1}$ and $\partial\bar{A}(T')$.

PROOF: By the induction hypothesis $\bar{K}_{i-1}$ is an orientable n-pseudomanifold. So is $\bar{A}(T')$. Let us denote $L = \bar{K}_{i-1} \cap \bar{A}(T')$ for notational convenience. L then is closed under nonempty subsets, and so is a complex. Let $x \cup Q \in L$. Then $x \in A(T)$ for some T, $|T| < d + i$, and $x \in A(T')$, $Q \subset Q_{T'}$. By assumption (ii), there exists $k \in T'$ such that $x \in A(T' \setminus \{k\})$. Let $y \in A(T' \setminus \{k\})$ contain x $(x \subset y)$, such that $T_y = T' \setminus \{k\}$. Then $x \cup Q \subset y \cup Q_{T'}$. Note $y \cup Q_{T'} \in L$. Furthermore $|y \cup Q_{T'}| = d + i + n - (d + i) = n$. Thus every element of L is a subset of an $(n-1)$-simplex of L.

Now let $x \cup Q$ be an $(n-1)$-simplex of L. From the preceding remarks, we know $T_x = T' \setminus \{k\}$ for some $k \in T'$, and $Q = Q_{T'}$. Let $x \cup Q \setminus \{\alpha\}$ be an $(n-2)$-simplex of L, and suppose $x \cup Q \setminus \{\alpha\} \cup \{\beta\}$ is an $(n-1)$-simplex of L, $\beta \neq \alpha$. We need to show that there is at most one choice of β. Clearly, $\alpha \notin Q$, so $\alpha \in x$. We have two cases:

Case I. $x \setminus \{\alpha\}$ is not full. Then since $x \setminus \{\alpha\} \cup \{\beta\}$ must be full, β is the unique element of K^0 such that $x \setminus \{\alpha\} \cup \{\beta\}$ is a $(d+i-1)$-simplex of $A(T' \setminus \{k\})$.

Case 2. $x \setminus \{\alpha\}$ is full. In this case $T_{x\setminus\{\alpha\}} = T' \setminus \{k\} \setminus \{j\}$ for some $j \neq k$, $j \in T'$.

Since $x \setminus \{\alpha\} \cup \{\beta\}$ must be a full $(d + i - 1)$-simplex, β is the unique element of K^0 such that $x \setminus \{\alpha\} \cup \{\beta\}$ is a $(d + i - 1)$-simplex of $A(T' \setminus \{j\})$.

Thus we see that L is a pseudomanifold of dimension $(n-1)$.

Our next task is to show that L is homogeneous. Let $x \cup Q_{T'}$, $y \cup Q_{T'}$ be distinct $(n-1)$-simplices in L. If $T_x = T_y$, then since $A(T_x)$ is homogeneous, there is a sequence $x = s_0, \ldots, s_k = y$ of neighbors such that each $s_j \in A(T_x)$, $j = 0,\ldots, k$. Then $x \cup Q_{T'} = s_0 \cup Q_{T_1}, \ldots, s_j \cup Q_{T'}, \ldots, s_k \cup Q_{T'}$ is a sequence of neighbors in L. Suppose then $T_x \neq T_y$. $T_x = T' \setminus \{j\}$, $T_y = T' \setminus \{k\}$, for some j, k, where $j \neq k$, $j \in T'$, $k \in T'$. Let $z \in A(T' \setminus \{j\} \setminus \{k\})$. Then there exists $\alpha, \beta \in K^0$ such that $z \cup \{\alpha\} \in A(T' \setminus \{j\})$, $z \cup \{\beta\} \in A(T' \setminus \{k\})$. Let $x = s_0, \ldots, s_p = z \cup \{\alpha\}$ be a sequence of neighbors in $A(T' \setminus \{j\})$, $z \cup \{\beta\} = t_0, \ldots, t_r = y$ a sequence of neighbors in $A(T' \setminus \{k\})$. Then the sequence $x \cup Q_{T'} = s_0 \cup Q_{T'}, \ldots, s_p \cup Q_{T'}$, $t_0 \cup Q_{T'}, \ldots, t_r \cup Q_{T'} = y \cup Q_{T'}$ is a sequence of neighbors in L. Thus L is homogeneous.

Next we show that $L \subset \partial \bar{K}_{i-1}$. Let $x \cup Q_{T'}$ be an $(n-1)$-simplex of L. Since $T_x = T' \setminus \{k\}$ for some $k \in T'$, we can write $x \cup Q_{T'} = x \cup Q_{T_x} \setminus \{q_k\}$. Any n-simplex of $\bar{K}_{i-1}$ is of the form $y \cup Q_{T_y}$ where

$|T_y| \leq d + i - 1$. Thus the unique n-simplex of $\bar{K}_{i-1}$ containing $x \cup Q_{T'}$ is $x \cup Q_{T_x}$, and hence $x \in \partial\bar{K}_{i-1}$. A similar argument shows that $L \subset \partial\bar{A}(T')$.

It only remains to show that L is orientable. Since $\bar{K}_{i-1}$ is orientable, $Or(\cdot)$ on $\bar{K}_{i-1}$ induces an orientation $Or(\cdot)$ on $L \subset \partial\bar{K}_{i-1}$. We need to show that this induced orientation is coherent. Let $\bar{x}, \bar{y} \in L$ be neighbors. Let us assign labels to elements of $\bar{K}_{i-1}$ as follows: For $v \in \bar{K}_{i-1}$, $v \notin \bar{x} \cup \bar{y}$, let $L(v) = 1$. We can write $\bar{x} = \{v_1,\ldots,v_n\}$, $\bar{y} = \{v_{n+1}, v_2, \ldots, v_n\}$ and let $L(v_1) = L(v_{n+1}) = 1$, $L(v_i) = i$, $i = 2,\ldots, n$. Let us do C-pivots on the C-ordering of $\bar{x}$. This will trace a path of simplices of $\bar{K}_{i-1}$, which if it has a right endpoint, the right endpoint will be $\bar{y}$. Furthermore, by the nature of our labelling function, all elements of the path will contain $\bar{x} \cap \bar{y}$. At least one element of $\bar{x} \cap \bar{y}$ will be an element of K^0, and since K is locally finite, the path will have a right endpoint. By the results of Section 6.2, $Or(\bar{x}) = -Or(\bar{y})$, thus establishing that $Or(\cdot)$ is coherent on L. ⊗

With Lemma 6.14 established, we can orient L using the induced orientation $Or(\cdot)$ from $\bar{K}_i$. Now let $\bar{x} = \{v_1,\ldots, v_n\}$ be a fixed ordered element of L. Since $\bar{x} \in \partial\bar{K}_{i-1}$, $\bar{x} \in \partial\bar{A}(T')$, there exist unique elements $\alpha, \beta \in \bar{K}^0$ such that $\{\alpha, v_1,\ldots, v_n\} \in \bar{K}_{i-1}$, $\{\beta, v_1,\ldots, v_n\} \in \bar{A}(T')$. Define $Or(\beta, v_1, \ldots, v_n) = -Or(\alpha, v_1, \ldots, v_n)$, and extend $Or(\cdot)$ to all of $\bar{A}(t')$ by using $\{\beta, v_1,\ldots, v_n\}$ as a "seed". This makes $\bar{A}(T')$ coherently oriented, and also $\bar{K}_{i-1} \cup \bar{A}(T')$ coherently oriented.

We can repeat this procedure for all $T' \in \mathfrak{I}_{i-1}$, since for any $S, T \in \mathfrak{I}_{i-1}$, $\bar{A}(S) \cap \bar{A}(T)$ contains no (n-1)-simplices or n-simplices, i.e. $\bar{A}(S)$ and $\bar{A}(T)$ share no common boundary.

Thus Step i, i = 1,..., m, of our procedure is executable and results in a coherent orientation of $\bar{K}_i$. Hence $\bar{K}_m = \bar{K}$ is orientable. We have just proved:

Theorem 6.15. Let $A(\cdot)$, $\mathfrak{I}$, N, K satisfy assumptions (i) and (ii) of this section. Then $\bar{K}$ is orientable.

6.4. Concluding Remarks

We first discuss ways to verify assumptions (i) and (ii) of Section 6.3. Assumption (ii) can be verified by a case by case analysis of elements of $\mathfrak{I}$, if need be. However, notice that if $\mathfrak{I}$ is closed under subsets, then assumption (ii) is satisfied. In all of the applica-ions of V-complexes discussed in this thesis, the only instance where $\mathfrak{I}$ is not closed under subsets occurs in Kuhn's algorithm for the Sperner Lemma, where

$$\mathfrak{I} = \{\emptyset, \{1\}, \{1,2\}, \ldots, \{1,2,\ldots,n-1\}\} .$$

and in this case, Assumption (ii) is satisfied.

Assumption (i) can be difficult to verify, in general. However, it can be shown that if an n-pseudomanifold J can be realized as a triangulation C of a set S in $\mathbb{R}^n$, then J is orientable. The

proof of this statement involves the use of determinants and the notion of an orientable piecewise-linear subdivided manifold, and as it is not central to our discussion, we omit it. All of the sets $A(T)$ in all of our applications are realizable as triangulations of sets in $\mathbb{R}^{|T|}$, and so every specific H-complex of this thesis is orientable.

A final remark concerns whether or not $\bar{K}$ can be realized as a triangulation in $\mathbb{R}^n$, where $n = |N|$. In Eaves [6], it is shown that for Shapley's algorithm, $\bar{K}$ can be realized as such a triangulation. In general, the complex K can be realized as a triangulation $\mathbb{R}^n$. The problem then becomes how to place "artificial" vertices q_i, $i = 1,\ldots,n$, in $\mathbb{R}^n$ in such a way that $\bar{K}$ can be realized as a triangulation. This is an open question.

CHAPTER VII

KNOWN VARIABLE-DIMENSION ALGORITHMS INTERPRETED ON V-COMPLEXES

7.0. Introduction

The original ideas behind the development of V-complexes came from reading the papers of van der Laan and Talman [24], Reiser [33], and Lüthi [31], who present variable-dimension algorithms for computing fixed-points or nonlinear complementarity points. In this chapter, we show that these and other variable-dimension algorithms can be interpreted as path-following on a V-complex.

In Section 1, we present fixed-point algorithms, namely those of Kuhn [44], Garcia [12], and van der Laan and Talman [23] and [24]. In Section 2, we present the algorithms of Lüthi [31] and Reiser [33] for the nonlinear complementarity problem. In Section 3, we present the algorithm of van der Laan and Talman [25] for equilibrium points in n-person game theory.

7.1. Fixed Point Algorithms

As was shown in Chapter V, Sperner's Lemma and Brouwer's Fixed-Point Theorem are "equivalent," in the sense that one provides a quick proof of the other. The first variable-dimension algorithm known to the author is Kuhn's algorithm [44] for the Sperner Lemma which, as we have shown in Chapter V, can be used to compute approximate fixed-points.

Since we have already cast this algorithm as an instance of path-following on a V-complex, refer to Chapter V, Section 1, for the details.

Garcia's Algorithm

The second variable-dimension algorithm for computing fixed-points was Garcia's "hybrid" algorithm [12]. Our treatment of this algorithm draws heavily on the material in Chapter V, Section 3. Let $S^n = \{x \in \mathbb{R}^n \mid e^T x = 1,\ x \geq 0\}$, and let $N = \{1,\ldots,n\}$. Let $j \in N$ be fixed. Then construct C, $\mathfrak{I}$, $A(\cdot)$, and K, K^0, as in Chapter V, Section 3. Let $f: S^n \to S^n$ be a given continuous function. For each $(v,t) \in K^0$, let

$$L(v,t) = \min\{i \mid f_i(v) \geq v_i\}.$$

The algorithm consists of following the infinite path from $\emptyset \in B$; the proof that this path is infinite appears in Section 3 of Chapter V. Let $\{x_i\}_{i=0}^{\infty}$ be the simplices encountered in the path. There exists $\bar{T} \subset N$, $\bar{T} \neq \emptyset$, such that $L(x_i) = \bar{T}$ for infinitely many i. For each x_i such that $L(x_i) = \bar{T}$, choose a point $(s^i,t^i) \in x_i$, and let s^* be a cluster point of the s^i. By the continuity of $f(\cdot)$, $f_i(s^*) \geq s_i^*$ for all $i \in \bar{T}$. For all $i \notin \bar{T}$, $s_i^* = 0$, since for all k sufficiently large $t^k > 0$. Thus $f_i(s^*) \geq s_i^*$ for all i, which implies that $f(s^*) = s^*$, since $e^T f(s^*) = e^T s^* = 1$.

van der Laan and Talman's First Algorithm

In [24], van der Laan and Talman presented a variable-dimension algorithm for Sperner's Lemma and for computing fixed-points. The pivot

rules and a sample path for this algorithm appear in Chapter II, Section 2. Here we present the algorithm as a path-following procedure on a V-complex. Let $S^n = \{x \in \mathbb{R}^n | e^T x = 1,\ x \geq 0\}$ and let C be the Scarf-Hansen triangulation of S^n. Let $N = \{1,\ldots,n\}$ and let $\mathfrak{T} = \{T \subset N | T \neq N\}$. Let $w > 0$ be a fixed vertex of C, and let

$$Q = \begin{bmatrix} -1 & 0 & \cdot & \cdot & \cdot & & 0 & 1 \\ 1 & -1 & & & & & & 0 \\ 0 & 1 & \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & & & \cdot & -1 & 0 \\ 0 & \cdot & \cdot & \cdot & & 0 & 1 & -1 \end{bmatrix}$$

where Q is an $n \times n$ matrix. Let q^i denote the $i^{\underline{th}}$ column of Q. For each $t \in \mathfrak{T}$, let $A(T)$ be the pseudomanifold corresponding to the restriction of C to

$$\{x \in S^n | x = w + \sum_{i \in T} \lambda_i q^i,\ \lambda_i \geq 0\} .$$

Finally, let K be the pseudomanifold corresponding to C. It can be shown that K, $A(\cdot)$ and $\mathfrak{T}$ define a V-complex. See Figure 7.1.

Now let $L(\cdot)$ be a proper labelling (see Chapter IV, Section 1) of K^0, and let us examine the sets B and G. Clearly $\emptyset \in B$, since $A(\emptyset) = \{\emptyset, \{w\}\}$. Suppose $\emptyset \neq x \in B$. We have $T_x = L(x)$, and $x \in \partial' A(T_x)$. Let $x = \{v^1, \ldots, v^t\}$. Then $v^i_j = 0$ for some $j \in T_x$ and all $i = 1,\ldots,t$. Thus since $L(\cdot)$ is proper, $j \notin L(x)$, a contradiction. Therefore $B = \{\emptyset\}$.

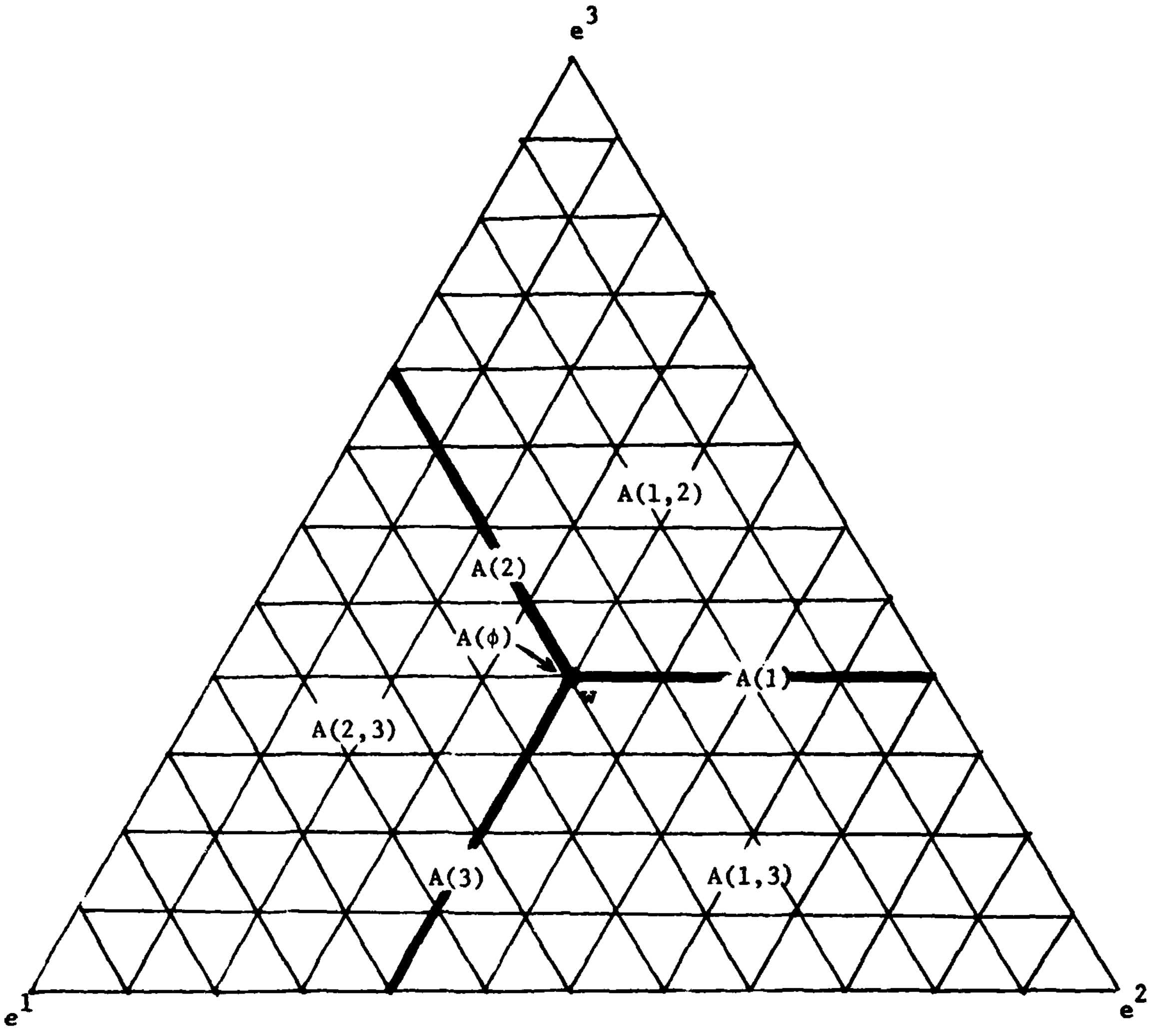

Figure 7.1

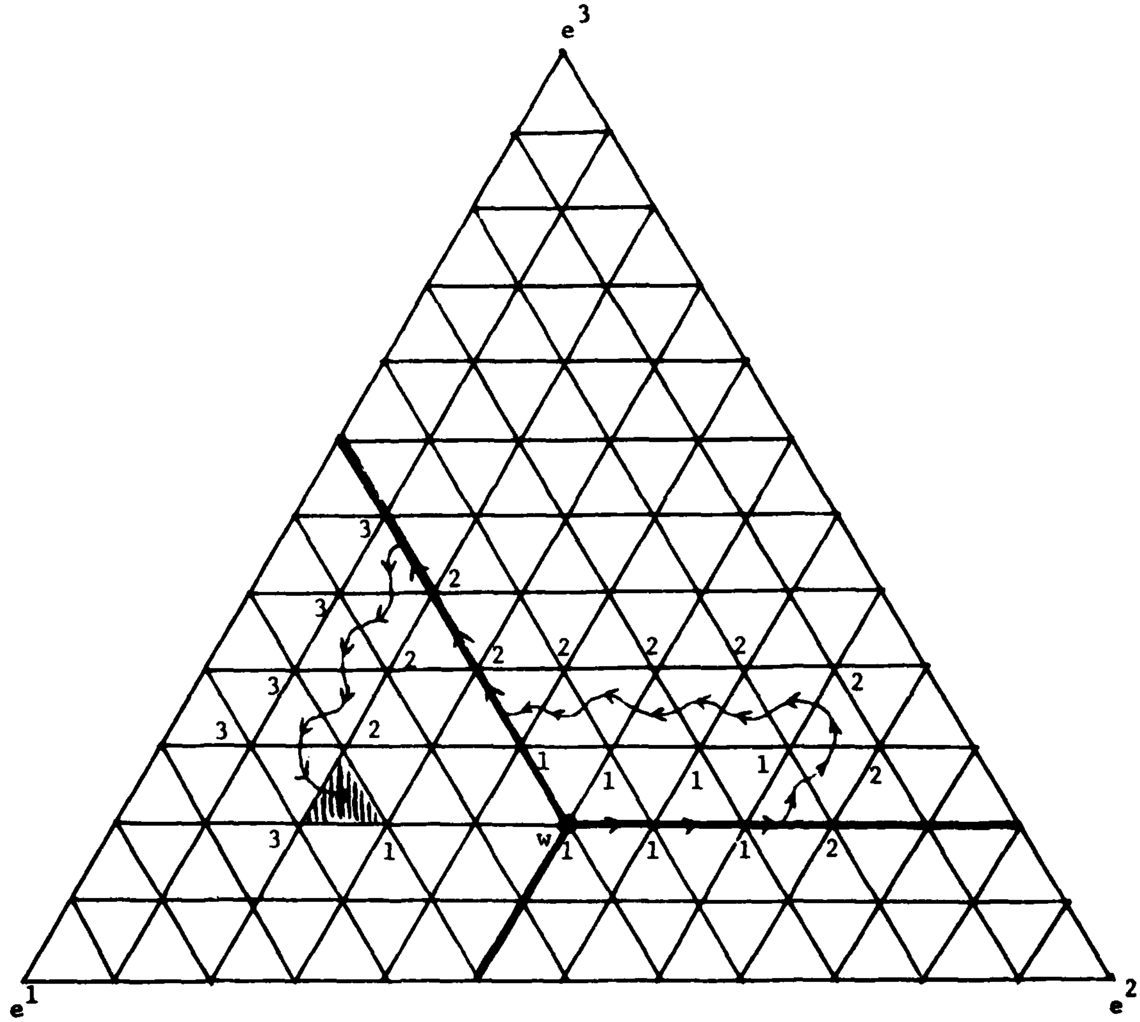

Figure 7.2

Suppose $x \in G$. Then by Definition of $\Im$, $L(x) = N$, and x is a completely-labelled simplex. The algorithm consists of following the path whose endpoint is $\emptyset \in B$. The other endpoint must be an element of G, and so is a completely labelled simplex. A sample path of this algorithm appears in Figure 7.2.

van der Laan and Talman's Second Algorithm

In [23], van der Laan and Talman presented a variable-dimension algorithm for computing fixed points on unbounded regions. An analysis of this algorithm as a V-complex is as follows:

Let K be the pseudomanifold corresponding to Kuhn's triangulation of $\mathbb{R}^n$, and let $f: \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Let Q be the $n \times (n+1)$ matrix

$$Q = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 \\ 0 & 1 & \cdot & & & & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & 0 & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -1 \end{bmatrix}$$

Also, let $w \in K^0$ be fixed. Let $N = \{1, \ldots, n+1\}$, and let $\Im = \{T \subset N \mid T \neq N\}$. Define $A(\emptyset) = \{\emptyset, \{w\}\}$, and for $\emptyset \neq T \in \Im$, define $A(T)$ to be the pseudomanifold corresponding to the restriction of Kuhn's triangulation to

$$\{x \in \mathbb{R}^n \mid x = w + \sum_{i \in T} \lambda_i q^i,\ \lambda_i \geq 0\} ,$$

where q^i is the $i^{\underline{th}}$ column of Q. It can be verified that $A(\cdot)$, $\mathfrak{I}$, and N define a V-complex. See Figure 7.3. Now let $L(\cdot): K^0 \to N$ be a labelling function defined by

$$L(v) = \begin{cases} i & \text{if } f_i(v) - v_i \geq f_k(v) - v_k \text{ for all } k \text{ and} \\ & \quad f_i(v) - v_i \geq 0 \\ & \\ n+1 & \text{if } f(v) - v < 0 \,. \end{cases}$$

Let us now examine B and G. Clearly $\emptyset \in B$, since $A(\emptyset) = \{\emptyset, \{w\}\}$. Also since $\partial' A(T) = \emptyset$ for all $T \neq \emptyset$, $B = \{\emptyset\}$. Also, by definition of $\mathfrak{I}$, $G = \{x \in K | L(x) = N\}$.

The algorithm consists of following the path from $\emptyset \in B$. If this path is finite, then its other endpoint is an element of G. The usual limiting argument shows that for sufficiently small diameter of the triangulation, any element of a simplex in G is an approximate fixed-point of $f(\cdot)$. In [23] and [32], sufficient conditions are given which guarantee the finiteness of the path from $\emptyset \in B$, thus assuring that an element of G, and hence an approximate fixed-point, is found. A sample path appears in Figure 7.4.

7.2. Algorithms for the Nonlinear Complementarity Problem

The nonlinear complementarity problem (NLCP) is as follows: let $f: \mathbb{R}^{n+} \to \mathbb{R}^n$ be given. Find $x^* \in \mathbb{R}^{n+}$ such that $f(x^*) \geq 0$ and $x^* f(x^*) = 0$. This problem, central to mathematical programming, arises in constrained optimization, game theory, and economic equilibrium theory. See, for example, Cottle and Dantzig [2], Eaves [3], Lemke [28] and [29], and Scarf [35].

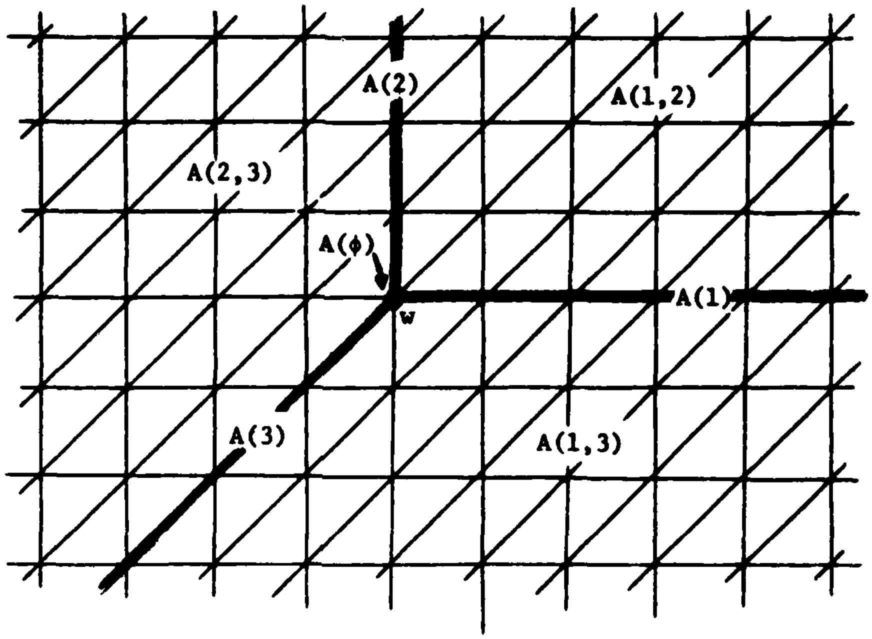

Figure 7.3

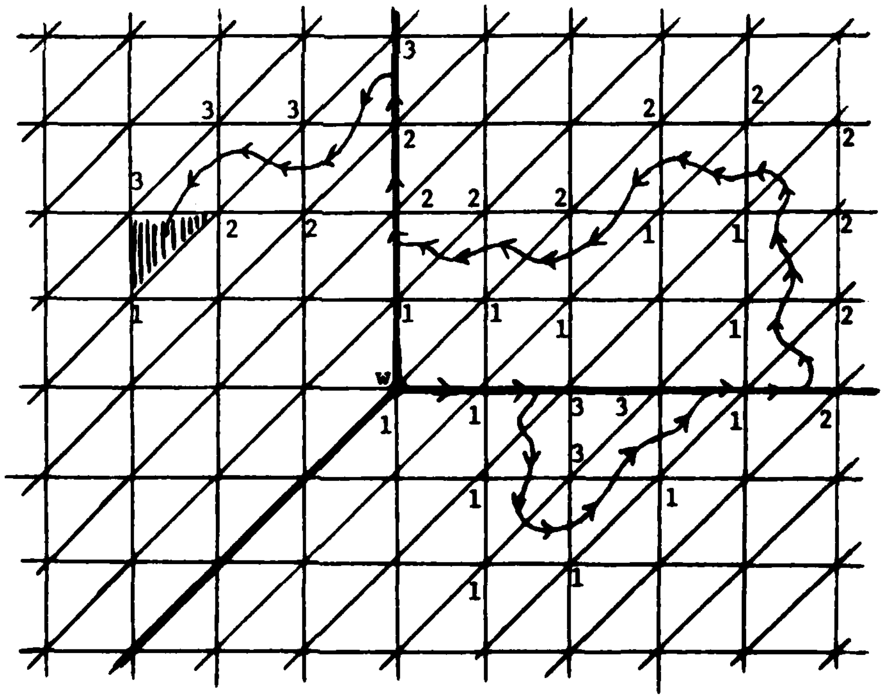

Figure 7.4

In [31], Lüthi presented a variable-dimension algorithm for the NLCP. In our presentation of his algorithm as an instance of path-following on a V-complex, we only treat the basic algorithm, and not the restart procedure.

Let $f: \mathbb{R}^{n+} \to \mathbb{R}^n$ be given, let C be a triangulation of $\mathbb{R}^{n+}$, and let K be the pseudomanifold corresponding to C. Let $N = \{1,\dots,n+1\}$, and let $\mathfrak{I} = \{T \subset N \mid n+1 \notin T\}$. Let $A(\emptyset) = \{\emptyset, \{0\}\}$, and for $\emptyset \neq T \in \mathfrak{I}$, define $A(T)$ to be the pseudomanifold corresponding to the restriction of C to the set

$$\{x \in \mathbb{R}^{n+} \mid x_i = 0 \text{ for all } i \notin T\} .$$

It can be verifed that $\mathfrak{I}$, N, and $A(\cdot)$ define a V-complex. See Figure 7.5.

Now let the labelling function $L(\cdot): K^0 \to N$ be defined as follows:

$$L(v) = \begin{cases} n+1 & \text{if } f(v) \geq 0 \\ \text{any } i \text{ such that } f_i(v) \leq f_j(v) & \text{for all } j \text{ if } f(v) \ngeq 0. \end{cases}$$

Let us now examine the sets B and G. Clearly, $\emptyset \in B$, since $A(\emptyset) = \{\emptyset, \{0\}\}$. Furthermore, for $T \in \mathfrak{I}$, $T \neq \emptyset$, $\partial' A(T) = \emptyset$, so therefore $B = \{\emptyset\}$.

Let $x \in G$. Then $L(x) = T_x \cup \{n+1\}$, by definition of N and $\mathfrak{I}$. Let s be an element of the real simplex corresponding x. Then if the diameter of x is sufficiently small, we have that $f(s) \gtrapprox 0$ (where "$\approx$" denotes approximately), since $(n+1) \in L(x)$. Suppose $s_i > 0$ for some i.

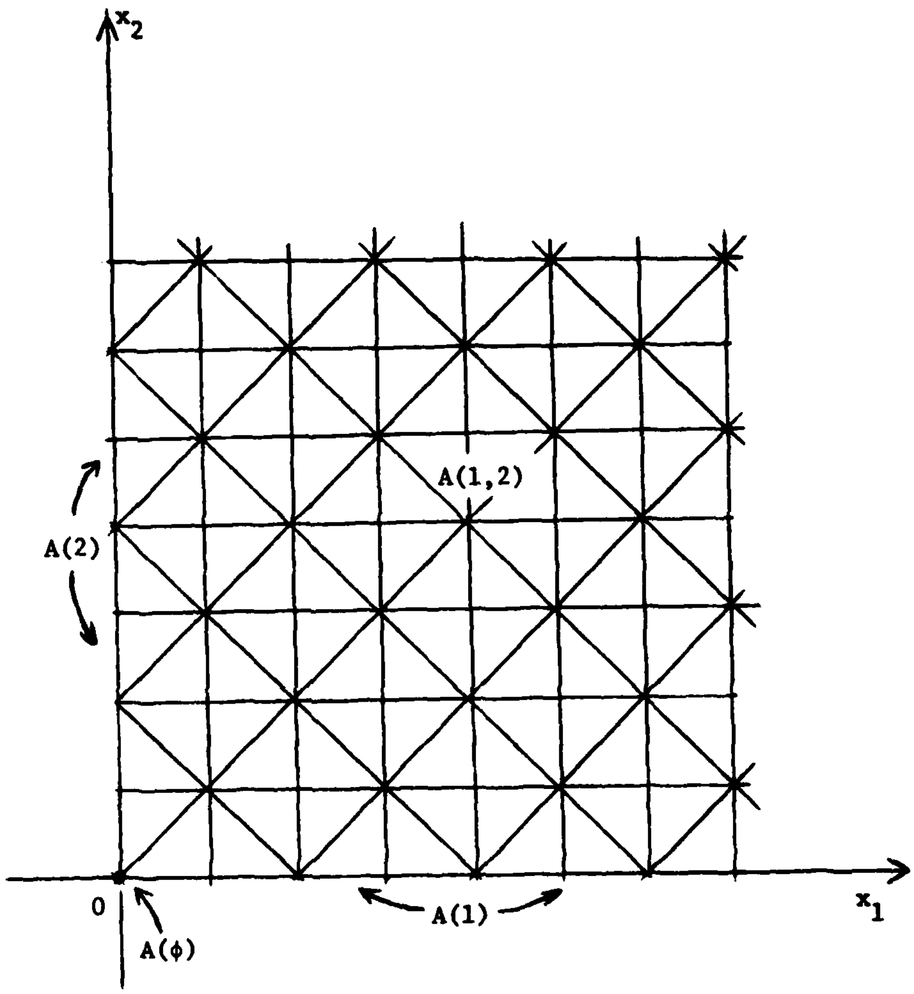

Figure 7.5

Then we must have $i \in T_x$. Thus $i \in L(x)$, and so $f_i(s) \lessapprox 0$, which implies $f_i(s) \approx 0$. Thus we have that $s \cdot f(x) \approx 0$, and so s solves the NLCP.

The algorithm consists of following the path from $\emptyset \in B$. If this path is finite, then any element of the terminal simplex $x \in G$ is an approximate solution to the NLCP. Conditions which guarantee the finiteness of the path are given in Kojima [19]. A sample path appears in Figure 7.6.

Reiser's algorithm

In [33], Reiser presented a variable-dimension algorithm for the NLCP, and first introduced the use of negative-valued integer labels. Reiser's exact algorithm is slightly different from that which appears in Chapter II, Section 1. The analysis of Reiser's algorithm as an instance of path-following on a V-complex is as follows:

Let $f: \mathbb{R}^{n+} \to \mathbb{R}^n$ be continuous. Let K be the pseudomanifold corresponding to Kuhn's triangulation of $\mathbb{R}^{n+}$, and let $N = \{1, \ldots, n, -1, \ldots, -n\}$. Let $w > 0$ be a fixed element of K^0. Let $\mathfrak{J} = \{T \subset N \mid i \in T \text{ implies } -i \notin T\}$ and define $A(\emptyset) = \{\emptyset, \{w\}\}$. For $\emptyset \neq T \in \mathfrak{J}$, define $A(T)$ to be the pseudomanifold corresponding to the restriction of Kuhn's triangulation to

$$\{x \in \mathbb{R}^{n+} \mid j \notin T,\ -j \notin T \text{ implies } x_j = w_j, \text{ and } j \in T \text{ implies } j \cdot (x-w)_{|j|} \leq 0\}.$$

See Figure 7.7. It can be verified that K, $A(\cdot)$, N, and $\mathfrak{J}$ define a V-complex. Let $L(\cdot): K^0 \to N$ be the labelling function

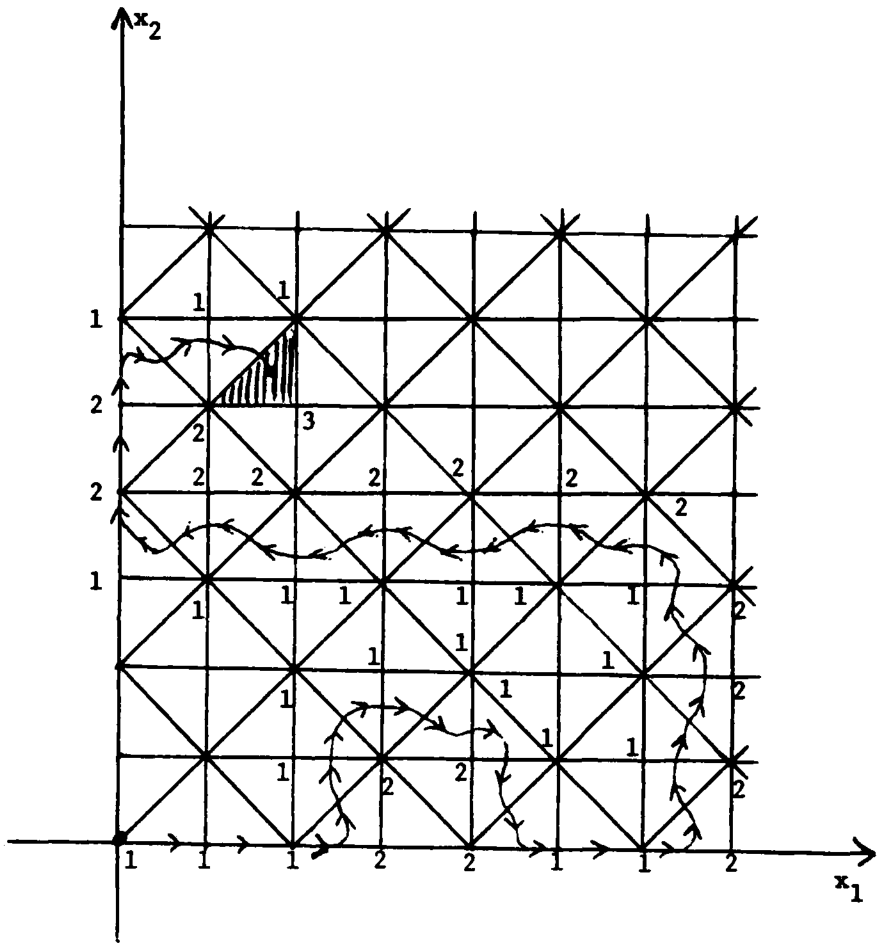

Figure 7.6

$$L(v) = \begin{cases} i & \text{if } \min\limits_{j=1,\ldots,n} f_j(v) + \max\limits_{v_j>0} f_j(v) > 0 \\ & \text{and } f_i(v) \geq f_j(v) \text{ for any } v_j > 0, \text{ and } v_i > 0. \\ -i & \text{if } \min\limits_{j=1,\ldots,n} f_j(v) + \max\limits_{v_j>0} f_j(v) \leq 0 \\ & \text{and } f_i(v) \leq f_j(v) \text{ for any } j = 1,\ldots, n \\ 0 & \text{if } v = 0,\ f(v) \geq 0 . \end{cases}$$

If $L(v) = 0$, then $v = 0$ solves the NLCP and we are done. So we assume that $L(0) \neq 0$. In the above labelling function, choose the least index i in case of ties.

Let us now examine the sets B and G. Clearly $\emptyset \in B$, since $A(\emptyset) = \{\emptyset, \{w\}\}$. Suppose $\emptyset \neq x \in B$. Then $x \in \partial' A(T_x)$. Thus for all $v \in x$, there is some $i \in T_x$ such that $v_i = 0$. But then we cannot have $L(v) = i$, so that $L(x) \neq T_x$. This contradicts the definition of B. Therefore $x \notin B$ and so $B = \{\emptyset\}$.

Now let $x \in G$. Then there are vertices $v', v'' \in x$ such that $L(v') = -L(v'')$. In [33], Reiser shows that for sufficiently small diameter of C, v' or v'' is an approximate solution to the NLCP.

Reiser's algorithm consists of following the path from $\emptyset \in B$. If this path is finite, it must terminate with an element x of G. In [33], Reiser gives sufficient conditions on f that guarantee that the path from $\emptyset \in B$ is finite. See Figure 7.8 for a sample path.

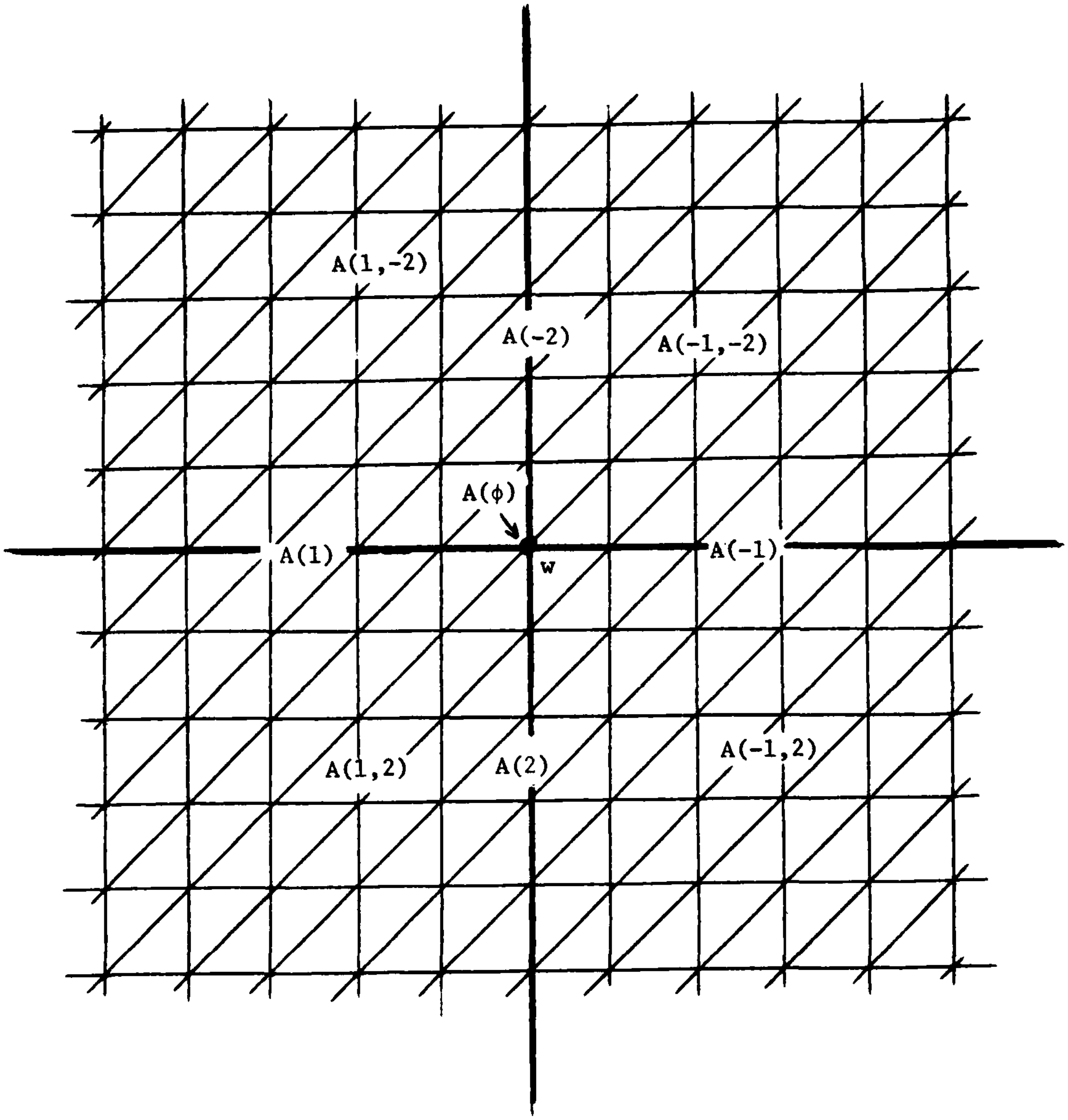

Figure 7.7

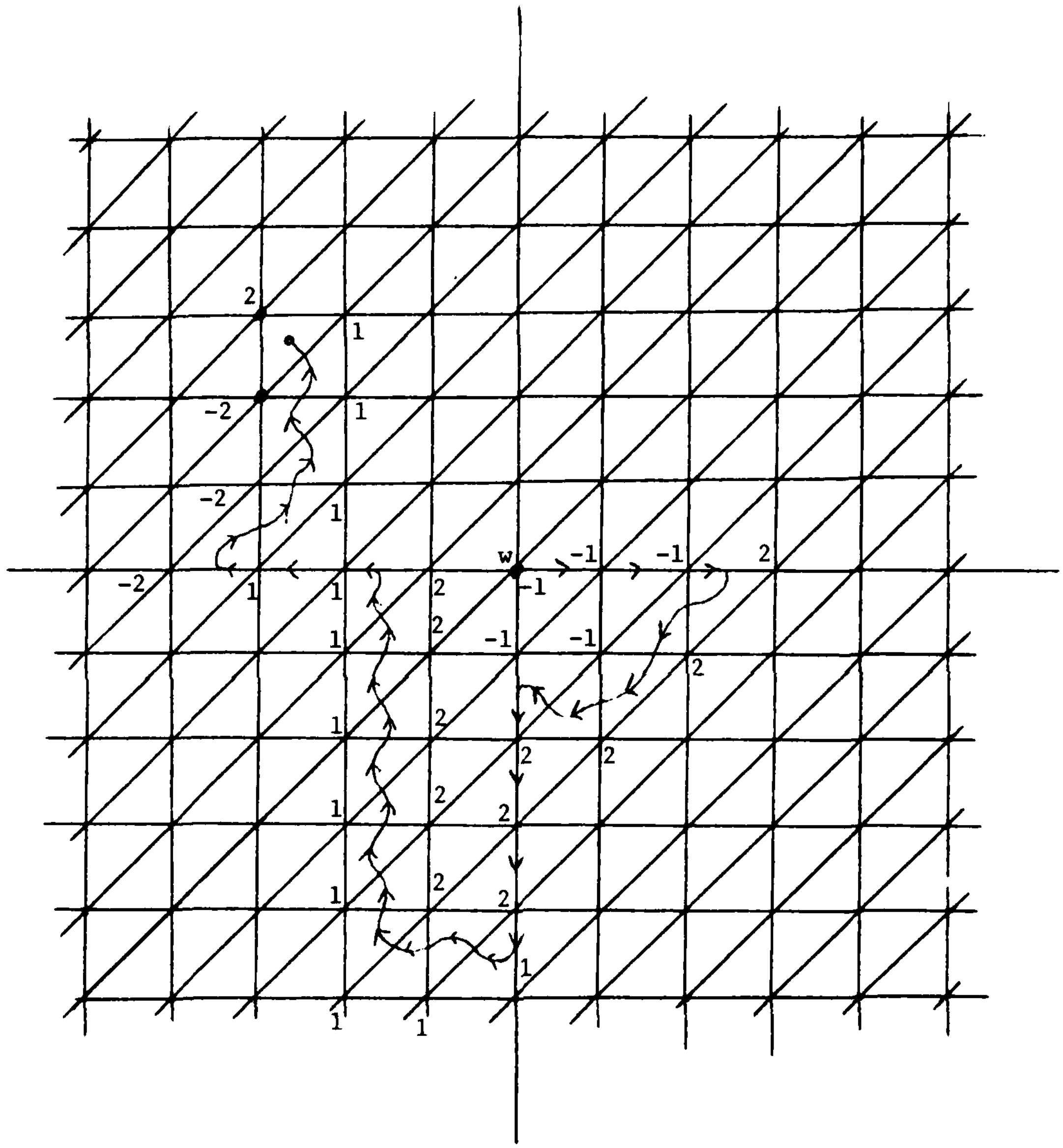

Figure 7.8

7.3. Algorithms for Equilibrium Points in n-Person Noncooperative Game Theory

An n-person noncooperative (finite) game can be described as follows: There are N players $(1 < N < \infty)$, each of whom have m_n pure strategies, $n = 1, \ldots, N$. Let $\bar{n} = \sum_{n=1}^{N} m_n$. The given quantities

$$a(n;\, i_1, \ldots, i_N)\ , \qquad i_j = 1, \ldots, m_n; \quad n = 1, \ldots, N; \quad j = 1, \ldots, N$$

represent the loss per play to player n if player j plays his/her i_jth pure strategy. These quantities are assumed to be positive without loss of generality. We denote the strategy vector

$$\underline{i} = (i_1, \ldots, i_N) \in I_1 \times \cdots \times I_N\ ,$$

where

$$I_n = \{1, \ldots, m_n\}, \qquad n = 1, \ldots, N\ .$$

Let

$$S^n = \{x \in \mathbb{R}^{m_n} \,|\, e^T x = 1,\ x \geq 0\}$$

be the set of mixed strategies for player n, $n = 1, \ldots, N$. Furthermore, let $S = S^1 \times \cdots \times S^N$. S, then, is the set of all mixed strategies for all players. Let $I = I_1 \times \cdots \times I_N$. Define

$$f(n;i,x) = \sum_{\substack{\underline{i} \in I \\ i_n = i}} a(n;\, i_1, \ldots, i_{n-1}, i, i_{n+1}, \ldots, i_N) \prod_{\substack{k=1 \\ k \neq n}}^{N} x(k, i_k)\ ,$$

where $x(k,j)$ is the coefficient of the $j^{\underline{th}}$ term of the $k^{\underline{th}}$ mixed strategy. $f(n;i,x)$ is the marginal loss to player n under the mixed strategy $x \in S$, for each $i \in I_n$.

A strategy vector $\bar{x}$ is a (Nash) equlibrium point if and only if $\bar{x}$ satisfies:

i) $\bar{x} \in X$

ii) $[f(n;i,\bar{x}) - h(n,\bar{x})]\, \bar{x}(n,i) = 0,\ n = 1,\ldots, N;\ i = 1,\ldots, m_n,$

where $h(n,\bar{x}) = \min\limits_{i \in I_n} f(n;i,\bar{x})$.

In [25], van der Laan and Talman present an algorithm that computes an approximation of $\bar{x}$. To do so, they also present a neat way to triangulate S. Before interpreting their algorithm as path-following on a V-complex, we first need to describe their triangulation.

Let $d_1, \ldots, d_N$ be fixed positive integers, and let $w > 0$ be a fixed element of S such that $d_n \cdot w(n,i)$ is integral for all $n = 1,\ldots, N$, $i = 1,\ldots, m_n$. Let

$$Q_n = \frac{1}{d_n} \begin{bmatrix} -1 & 0 & \cdot & & \cdot & \cdot & 0 & 1 \\ 1 & -1 & \cdot & & & & & 0 \\ 0 & 1 & \cdot & \cdot & & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & & -1 & 0 \\ 0 & \cdots & 0 & & & & 1 & -1 \end{bmatrix}$$

where Q_n is an $(m_n) \times (m_n)$ matrix. Let Q be defined by:

$$Q = \begin{bmatrix} Q_1 & & & \bigcirc \\ & \cdot & & \\ & & \cdot & \\ \bigcirc & & & \cdot \\ & & & Q_N \end{bmatrix}$$

Furthermore, let q_j be the $j^{\underline{th}}$ column of Q. Let v^0 be any point in the affine hull of S, such that $d_n \cdot v(n,i)$ is integral for $n = 1,\ldots, N$, $i = 1,\ldots, m_n$. Choose indices $k_1, \ldots, k_N$ such that $k_n \in I_n$, $n = 1,\ldots, N$. And let π be a permutation of $I \setminus \{k_1,\ldots, k_N\}$. Define for $j = 1,\ldots, \bar{n}-N$,

$$v^j = v^{j-1} + q_{\pi(j)} \; .$$

Then it can be shown that $\langle v^0,\ldots, v^{\bar{n}-N}\rangle$ is an $(\bar{n}-N)$-dimensional real simplex in the affine hull of S. The collection of all such simplices on S can be shown to be a triangulation of S. Call this triangulation C.

Let us now set up our V-complex related to S. Let

$$\tilde{N} = \{(n,i) \mid i \in I_n, \; n = 1,\ldots, N\}.$$

Let $\mathfrak{I} = \{T \subset \tilde{N} \mid \text{for each } n, \; \{(n,i) \mid (n,i) \in T\} \neq \{(n,1),\ldots, (n,m_n)\},$ $n = 1,\ldots, N\}$. Define $A(\emptyset) = \{\emptyset, \{w\}\}$. For each $T \in \mathfrak{I}$, $T \neq \emptyset$, define $A(T)$ to be the pseudomanifold corresponding to the restriction of C to the set

$$\{x \in S \mid x = w + \sum_{(n,i)\in T} \lambda_{(n,i)} q_{(n,i)}, \; \lambda_{(n,i)} \geq 0\} \; ,$$

where $q(n,i)$ is the $(\sum_{j=1}^{n-1} m_j + i)^{\underline{th}}$ column of Q.

It can be shown that this restriction is well-defined, and that K, $\bar{N}$, $\Im$, and $A(\cdot)$ define a V-complex, where K denotes the psueodmanifold corresponding to C.

We now define a labelling function $L(\cdot): K^0 \to \bar{N}$, by the rule: $L(v) = (n,i)$ if (n,i) is the lexicographic least index with $x(n,i) > 0$ and $f(n,i,v) - h(n,v) \geq f(m,j,v) - h(m,v)$ for all other $(m,j) \in \bar{N}$. It can be shown that this labelling rule is well-defined.

Let us now examine the sets B and G. Clearly $\emptyset \in B$, since $A(\emptyset) = \{\emptyset, \{w\}\}$. Suppose, $\emptyset \neq x \in B$, where $x = \{v^1, \ldots, v^t\}$ for some $t > 0$. Then $L(x) = T_x$ and $x \in \partial' A(T_x)$. Thus, for some $(h,i) \in T_x$, $v^j_{(h,i)} = 0$ for all $j = 1, \ldots, t$. But then, by definition of $L(\cdot)$, $(n,i) \notin L(x)$, contradicting the fact that $L(x) = T_x$. Thus $B = \{\emptyset\}$.

Since B contains only one element, G must contain an odd number of elements, by Corollary 3.11. Let $x \in G$. Then, for some $n \in \{1, \ldots, N\}$, $L(x) \supset \{(n,1), \ldots, (n,m_n)\}$. As is pointed out in [25], any element of x represents an approximate equilibrium point, for sufficiently small diameter of C. van der Laan and Talman's algorithm consists of following the path from $\emptyset \in B$ to its other endpoint, an element of G.

7.4. Concluding Remarks

In this chapter, we have shown that a number of variable-dimension algorithms can be formulated as path-following schemes on a V-complex. There are two noteworthy variable-dimension algorithms whose formulation in the context of a V-complex appears doubtful. These are the game-theory algorithm of Garcia, Lemke, and Lüthi [13], and the algorithm of van der Heyden [16] for the linear complementarity problem.

The algorithm of Garcia, Lemke, and Lüthi [13] for equilibrium points in noncooperative n-person games is truly a variable-dimension algorithm. However, it uses neither integer nor vector labels, but rather uses a sequence of labels, called p-labels, in its execution. The label function varies as a function of the state of the algorithm. It is the lack of a unique labelling function in the algorithm which makes it difficult to interpret the algorithm on a V-complex.

Van der Heyden's algorithm [16] for the linear complementarity problem is also a variable-dimension algorithm. In this algorithm, the dimension of the "simplex" (actually the set of relevent non-basic variables) can jump by more than one-dimension between two "adjacent" states. This makes the interpretation of the algorithm on a V-complex very doubtful as well.

With the exception of these two algorithms, all variable-dimension simplicial algorithms that have come to my attention can be interpreted as path-following schemes on a V-complex.

CHAPTER VIII

CONCLUDING REMARKS

8.0. Introduction

In this chapter, we make some summary remarks and conclusions regarding V-complexes, H-complexes, and their extensions. To begin with, let us summarize the development of the last five chapters of this dissertation. In Chapter III, we axiomatically defined a V-complex. We then showed how to "lift" a V-complex into an n-dimensional pseudo-manifold called an H-complex. With a labelling function specified, we showed how to do path following on V-complexes and equivalently on H-complexes. In Chapter IV, we used V-complexes to prove a variety of results in combinatorial topology, some new, some old. In Chapter V, we applied these results to prove a number of existence theorems in the mathematical theory of operations research, where we interpret operations research broadly to include optimization, game theory, and fixed-point theory, among other fields. In Chapter VI, we developed an orientation theory and an associated result on path orientation. Finally, in Chapter VII, we show how most variable-dimension simplicial algorithms can be interpreted as path-following algorithms on a V-complex.

8.1. Vector Labelling

In most simplicial algorithms, vector-valued rather than integer-valued labelling functions are used. A typical algorithm traces a path of

zeroes of a piecewise-linear (PWL) map induced by the labelling function. When integer labels are used, the algorithm can still be interpreted as tracing a path of zeroes of a PWL map, but is more naturally interpreted via the "ghost story" as stepping between "almost-completely-labelled" simplexes of an n-pseudomanifold.

In this dissertation, we have restricted ourselves, for the sake of clarity, to integer-labelling functions. However, the extension of path-following on a V-complex to vector-valued labels is not difficult. The following summary remarks show briefly how to extend our results to vector labelling:

Let K be a V-complex, and let $\bar{K}$ be its associated H-complex. Let $\bar{Q} = \{q_1, \ldots, q_n\}$ be the artifical vertices used in $\bar{K}$. Assume $\bar{K}$ can be realized as a triangulation C of some n-dimensional set S in $\mathbb{R}^p$ where $p \geq n$, with vertices $\bar{K}^0$. Let $\ell_1, \ldots, \ell_n$ be pre-specified (n-1)-vectors in $\mathbb{R}^{n-1}$ such that the system

$$\sum_{i=1}^{n} \lambda_i \ell_i = 0 , \qquad \lambda_i \geq 0, \; i = 1, \ldots, n$$

$$e^T \lambda = 1$$

has a unique solution. Then let $L(\cdot): K^0 \to \mathbb{R}^{n-1}$ be any vector labelling function, and extend $L(\cdot)$ to $\bar{K}^0$ by the rule

$$L(q_i) = \ell_i , \qquad i = 1, \ldots, n.$$

Then we can perform PWL path-following on $\bar{K}$ much as in Eaves [6]. Of course, we need to assume a regularity condition on $L(\cdot)$ or specify

a perturbation scheme in order to guarantee our path is well-behaved (i.e. no bifurcation takes place), but the essential ingredients are all as stated above.

8.2. Kojima's Work

In [20], Kojima has presented an interpretation of variable-dimension algorithms as PWL path-following on a special type of subdivided manifold with primal and dual elements. His work is neither a generalization nor a special case of our own. In his approach, the notion of triangulation is not used, but rather he works with subdivided manifolds. His dual subdivided manifolds correspond loosely with our simplices that are subsets of $\bar{Q}$, but his dual manifolds are not necessarily simplices. In this sense, his work is more general. In another sense, our work is more general, since we are not restricted to simplicial structures that can be imbedded in real n-dimensional space, i.e. our results depend only on pseudomanifolds.

8.3. V-Complex Topology

The structure of a V-complex, as we have seen, is a tool that is useful in other mathematical systems. However, its straightforward axioms make it somewhat interesting in its own right. One is led naturally to ask, "What kinds of sets can be realized as triangulations derived from V-complexes?" Although we have no definitive answer to this question, some remarks are in order.

Consider a 2-pseudomanifold K that is combinatorially equivalent to the Möbius strip (see Chapter I, Section 3). After many attempts, we have been unable to design a nontrivial V-complex with K as the underlying complex, where by nontrivial we mean a V-complex such that $\mathfrak{S}$ contains more than one element.

Furthermore, we have been unable to construct a nontrivial V-complex associated with the two-dimensional torus.

Of course, both the torus and the Möbius strip are non-convex sets. This naturally leads to the question as to whether or not any nonconvex set can be realized as a nontrivial V-complex. This question is resolved in Freund [10], where we present a non-trivial V-complex associated with the n-sphere, a non-convex set for $n > 0$.

A final question is, "Under what circumstances can the (n-pseudomanifold) H-complex $\bar{K}$ be realized in $\mathbb{R}^n$ as a triangulation?" The H-complexes $\bar{K}$ associated with the V-complexes in this dissertation are all realizable in $\mathbb{R}^n$, but this by no means answers the question.

8.4. Applications to Polyhedra

Associated with a real n-dimensional polytope is an (n-1)-pseudomanifold derived from its boundary (see Adler and Dantzig [1], for example). Researchers in linear programming and combinatorial optimization have studied the structures of these pseudomanifolds in relation to the Hirsch conjecture on the diameter of polytopes and other problems as well.

Although we have not applied V-complexes to polyhedra, there appears to be potential use for V-complexes in the combinatorial study of polyhedra and their related pseudomanifolds.

8.5. More on the Combinatorial Lemmas

As a final note, we make some remarks regarding the combinatorial lemmas of Chapter IV. As is summarized at the end of Chapter V, the Generalized Sperner Lemma provides a direct proof of both Sperner's Lemma and Scarf's dual lemma. These latter two results provide a direct proof of Brouwer's Fixed-Point Theorem, and vice versa. However, whereas Lemma 4.6 and Kuhn's Lemma both imply Brouwer's Theorem, we have been unable to use Brouwer's Theorem to prove either result. In [21], Kuhn proves a weaker version of his lemma by appeal to Brouwer's Theorem. A natural question, still unanswered, is "Is there a similarly weaker version of Lemma 4.6 that is implied by Brouwer's Theorem?"

In Chapter V, we showed how combinatorial lemmas on the simplex and the cube can be used to prove Brouwer's Theorem. Are there combinatorial results on other polyhedra that prove Brouwer's Theorem? All of the combinatorial results on the simplex and cube are derivable by appeal to a V-complex. On other polyhedra, what sorts of V-complexes (associated with the polyhedra) could give rise to new combinatorial results?

Although we have no answers to these questions, we are confident that further study may give partial or complete answers, and look forward to the possibilities that new research can offer.

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SOL 80-11: VARIABLE-DIMENSION COMPLEXES WITH APPLICATIONS
by Robert M. Freund

In the past few years, researchers in fixed-point methods have developed a number of variable-dimension simplicial algorithms. These algorithms are shown to be specific realizations of pivoting methods on a V-complex. The concept of a V-complex is also used to give new and constructive proofs of a variety of known theorems in combinatorial topology and mathematical programming. Finally, V-complexes give rise to new theorems in complementarity theory and combinatorial topology, including a generalization of the Sperner Lemma, a covering theorem on the the simplex, and a new combinatorial lemma on the n-cube.